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Non-Gaussianity of large-scale CMB anisotropies: a non-perturbative approach

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Chapter 1

Introduction

One of the basic ideas of modern cosmology is represented by the *inflationary paradigm*. It is believed that there was an early period in the history of the universe, well before the epoch of nucleosynthesis, during which the universe expansion was accelerated.

By means of this simple kinematical feature, inflation is able to solve the problems of the Hot Big Bang model. But inflation has another very attractive feature, which today is regarded as the most important aspect of it: it can generate the primordial fluctuations which are the seeds for the inhomogeneities in the Large Scale Structure (LSS) and for the anisotropies in the Cosmic Microwave Background (CMB) that we observe today [1]. In the inflationary picture, the primordial density and gravity-wave perturbations are created from quantum fluctuations "redshifted" out of the horizon, where they remain "frozen".

Despite the simplicity of the inflationary paradigm, the mechanism by which the expansion is attained and the scenario of generation of the cosmological perturbations are not yet established. It is therefore of extreme importance to identify and compute the observable quantities that will permit to constrain the various models of inflation.

In particular, the deviation from a Gaussian statistics in the CMB temperature anisotropies is an observable capable to discriminate among different scenarios of generation of the primordial perturbations during the inflationary epoch. On the contrary of other observable quantities in cosmology, non-Gaussianity is an intrinsic non-linear effect, but its measurement has become possible with a new generation of satellite-based CMB experiments (*WMAP* and *Planck*).

In the present work we study the contributions to the non-linear temperature anisotropies on large scales, in order to give a prediction for the bispectrum of CMB temperature anisotropies which is the main statistical tool sensitive to the level of non-Gaussianity.

After a review of the inflationary paradigm and of cosmological perturbation theory up to second order, we discuss in detail the production of the primordial curvature perturbation up to second order, in the standard scenario of single-field slow-roll inflation and in other two alternative scenarios that have been proposed recently, namely the curvaton and inhomogeneous reheating scenario.

Then, we will follow the evolution of non linearities in the radiation and matter dominated epochs until recombination, by exploiting the conservation on large scales of the gauge-invariant curvature perturbation, and we study the transfer of these non linear perturbations to the temperature anisotropies on large scales, through the second order Sachs-Wolfe effect.

A non-perturbative approach is used to obtain an expression for the temperature fluctuations on large scales and the evolution equations, from which we can easily recover the usual perturbative results: this constitutes the original part of this work. Within this formalism we compute, using a functional integral tecnique, the bispectrum and trispectrum of CMB anisotropies including only the Sachs-Wolfe effect.

Finally, in order to include all the relevant contributions to the large-scale temperature anisotropies at second order, we study the early and late integrated Sachs-Wolfe effect, and the contribution from second order tensor modes. This way, we are able to compute the second-order analog of the radiation transfer function on large scales, to have a definite theoretical prediction for the level of non-Gaussianity in CMB anisotropies that includes both the primordial non-Gaussianity and all the additional non-linear contributions.

Chapter 2

Inflation

2.1 Introduction

In this Chapter we present the inflationary paradigm [2, 3] . We start by showing some unsatisfactory shortcomings of the Hot Big Bang model, then we discuss slow-roll inflation as the simplest model to obtain an accelerated expansion of the universe, and we see how inflation elegantly solves the aforementioned problems.

2.2 Shortcomings of the Hot Big Bang Model

The Hot Big Bang model has proven a successful theory to explain the evolution of our universe from a very hot and dense state, after the initial singularity. Its observationally tested predictions are the universe expansion, the light elements abundance, the origin of the CMB and the galaxy formation and evolution. However, there are a number of problems that cannot be given an adequate explanation in the context of the model. We will briefly list them and then show how they are solved by means of the inflationary paradigm.

2.2.1 Horizon problem

The concept of particle horizon

Photons travel on null paths characterized by $dr = \frac{dt}{a} = d\tau$, where a is the scale factor of the universe; the physical distance a photon could have traveled until time t is called the *particle horizon*, and is given by

$$R_H(t) = a(t) \int_0^t \frac{dt'}{a(t')} \, . \tag{2.2.1}$$

An observer at time t is able to exchange signals with all other observers in the universe only if the integral in eq. (2.2.1) diverges, and in this case the FRW metric is conformally related to all of Minkowski spacetime.

If on the other hand the integral converges, the FRW metric will be conformally related only to a portion of Minkowski spacetime, and not all the observers are in causal contact with one another.

Now, if we have the equation of state $P = w\rho$, $a(t) \propto t^{\frac{2}{3(1+w)}} = t^n$ with $n < 1$, and

$$R_H(t) = \frac{t}{1-n} = \frac{n}{1-n} H^{-1} \sim H^{-1} \,. \tag{2.2.2}$$

So the distance to the horizon is finite in the standard cosmology when considering matter or radiation, and in general a fluid with $P > 0$.

Incidentally we note that in this case $R_H(t) \simeq H^{-1}$ up to numerical factors: this is the reason why cosmologists use the terms horizon and Hubble radius interchangeably. In inflationary models, however, when $w < -1/3$, the horizon grows with respect to the Hubble radius: in a de-Sitter space, this growth is exponential.

A given physical length scale λ is within the horizon if $\lambda < H^{-1}$, and it is outside the horizon if $\lambda > H^{-1}$. In terms of the corresponding comoving wavenumber $k = a\frac{2\pi}{\lambda}$ we will have the following rule:

$$\frac{k}{aH} \ll 1 \implies \text{scale } \lambda \text{ outside the horizon, no causality} \tag{2.2.3}$$

$$\frac{k}{aH} \gg 1 \implies \text{scale } \lambda \text{ within the horizon, causality} \,. \tag{2.2.4}$$

Horizon problem

According to the standard cosmology, photons decoupled from the matter components at about $z \simeq 1100$, corresponding to $180{,}000(\Omega_0 h^2)^{-\frac{1}{2}}$yrs. From the *last-scattering surface* onwards photons free-stream to us, and we see them as the cosmic microwave background (CMB), whose spectrum is consistent with that of a black-body at a temperature of $2.726 \pm 0.001°$K, as measured by the $FIRAS$ instrument on the $COBE$ satellite.

Now, the length corresponding to our present Hubble radius at the time of last-scattering was

$$\lambda_H(t_{LS}) = R_H(t_0)\frac{a(t_{LS})}{a(t_0)} = R_H(t_0)\frac{T_0}{T_{LS}} \,. \tag{2.2.5}$$

During the matter dominated period, the Hubble length has decreased as

$$H^2 \propto \rho_m \propto a^{-3} \propto T^3 \tag{2.2.6}$$

and so, at last-scattering,

$$H_{LS}^{-1} = R_H(t_0)\left(\frac{T_{LS}}{T_0}\right)^{-\frac{3}{2}} \ll R_H(t_0) \,, \tag{2.2.7}$$

and we see that the length corresponding to our Hubble radius was much larger than the horizon at that time.

The volumes corresponding to these two scales are

$$\frac{\lambda_H^3(t_{LS}}{H_{LS}^3} = \left(\frac{T_{LS}}{T_0}\right)^{-\frac{3}{2}} \simeq 10^6 \,, \tag{2.2.8}$$

so there were about 10^6 causally disconnected regions within the volume that now corresponds to our horizon, and from which we receive photons.

This poses a problem: it is difficult to see how photons that were not in causal contact can have nearly the same temperature to a precision of 10^{-5}, as CMB experiments tell us. Moreover, we see that photons not in causal contact at the last-scattering surface have small anisotropies of the same order of magnitude.

2.2.2 Flatness (or age) problem

From the Friedmann equations we have that

$$(\Omega - 1) = \frac{K}{a^2 H^2} \tag{2.2.9}$$

where K is the curvature and $\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G}{3H^2}\rho$ is the density parameter. If the Universe is exactly spatially flat, then $\Omega = 1$ at all times; if, however, there is even a small amount of curvature, Ω has a non trivial time dependence. In the case of radiation domination $H^2 \propto \rho_\gamma \propto a^{-4}$, and

$$\Omega - 1 \propto \frac{1}{a^2 a^{-4}} \propto a^2 \,; \tag{2.2.10}$$

in the case of matter domination $H^2 \propto \rho_m \propto a^{-3}$, and

$$\Omega - 1 \propto \frac{1}{a^2 a^{-3}} \propto a \,. \tag{2.2.11}$$

In both cases $(\Omega(t) - 1)$ decreases going backwards in time.

We know that today $(\Omega_0 - 1)$ is close to 0; at a very early time, say at the Planck time ($t_P \simeq 10^{-43}$s, $T_P \simeq 10^{19}$GeV) its value was:

$$\frac{|\Omega - 1|_{t_P}}{|\Omega - 1|_{t_0}} \sim \frac{a^2(t_P)}{a^2(t_0)} \sim \frac{T_0^2}{T_P^2} \sim \mathcal{O}(10^{-64}) \,. \tag{2.2.12}$$

where $T_0 \simeq 10^{-13}$GeV is the present day temperature of the CMB.

We see that the value of $(\Omega - 1)$ at early times has to be fine-tuned to values close to zero, without being exactly zero: even small deviations of Ω from 1 would have led the universe to collapse or cooling in a few Planck times, and this is the reason why this problem is also referred to as the "age" or "oldness" problem.

Let us see how the hypothesis of adiabatic evolution of the universe (that is, $S = \text{const}$) is connected to the flatness problem.

During a radiation dominated period we have

$$H^2 \simeq \rho_\gamma \simeq \frac{T^4}{M_P^2} \tag{2.2.13}$$

from which we deduce

$$\Omega - 1 = K\frac{M_P^2}{a^2T^4} = K\frac{M_P^2}{S^{\frac{2}{3}}T^2} \tag{2.2.14}$$

and under the hypothesis of adiabaticity

$$|\Omega - 1|_{t_P} \propto \frac{1}{S^{\frac{2}{3}}} \simeq 10^{-60} \tag{2.2.15}$$

that is, $(\Omega - 1)$ is so close to zero at early times because the total entropy is very large.

Therefore the flatness problem is a problem of understanding why the initial conditions correspond to a universe so close to spatial flatness; on the other hand, we see that this problem arises because of the hypothesis of adiabatic evolution.

2.3 Slow-roll inflation

2.3.1 Dynamics of the inflationary universe

We start by considering a homogeneous and isotropic universe, which is well described by the Friedmann-Robertson-Walker (FRW) metric:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\phi^2\right)\right] \tag{2.3.1}$$

where t is the cosmic time, r, ϑ, ϕ are the comoving (polar) coordinates, $a(t)$ is the scale factor of the universe and K is the curvature constant of constant time hypersurfaces.

The dynamics is given by the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.3.2}$$

where $G_{\mu\nu}$ is the Einstein tensor, G is Newton's constant and $T_{\mu\nu}$ is the energy-momentum tensor of matter.

In the case of metric (2.3.1), $T_{\mu\nu}$ describes a perfect fluid with energy density ρ and pressure P:

$$T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + P g_{\mu\nu} \tag{2.3.3}$$

where u_μ is the four-velocity of the observer.

The Einstein equations give the Friedmann equations:

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{K}{a^2}\,, \tag{2.3.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}\left(\rho + 3P\right), \tag{2.3.5}$$

where $H \equiv \frac{\dot{a}}{a}$ is the Hubble expansion parameter.

Equation (2.3.5) shows that a period of accelerated expansion takes place if

$$P < -\frac{\rho}{3}\,. \tag{2.3.6}$$

In particular, a period in which $P = -\rho$ is called a de Sitter phase. The energy continuity equation $u_\nu \nabla_\mu T^{\mu\nu} = 0$ reads

$$\dot{\rho} + 3H\left(\rho + P\right) = 0 \tag{2.3.7}$$

and from eq. (2.3.4), neglecting the curvature which is soon redshifted away as a^{-2}, we see that in a de Sitter stage $\rho = \text{const}$ and

$$H = H_I = \text{const}\,. \tag{2.3.8}$$

Solving eq. (2.3.5) we find that the scale factor grows exponentially:

$$a(t) = a_i e^{H_I(t-t_i)} \tag{2.3.9}$$

where t_i is the time when inflation starts.

It is useful to define the number of e-foldings as

$$N = \ln[H_I(t_f - t_i)] \tag{2.3.10}$$

which is a measure of the duration of the inflationary period.

2.3.2 The inflaton field

Now we show that the condition (2.3.6) can be attained by means of a simple scalar field φ, which we call the *inflaton*.

The action for a minimally coupled scalar field is given by:

$$\begin{aligned} S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R[g] + \mathcal{L}_\varphi\right] = \\ = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R[g] - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right] \end{aligned} \tag{2.3.11}$$

where g is the determinant of the metric, $R[g]$ is the Ricci scalar for the metric $g_{\mu\nu}$, $M_P^2 = \frac{1}{8\pi G_N}$ is the reduced Planck mass and $V(\varphi)$ is the potential of the scalar field.

The corresponding energy momentum tensor is

$$T_{\mu\nu} = -2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L} = \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}\left[-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - V(\varphi)\right]. \quad (2.3.12)$$

By varying the action with respect to φ we obtain the equation of motion

$$\frac{1}{\sqrt{-g}}\,\partial_\nu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\mu\varphi\right) = \frac{\partial V}{\partial\varphi} \quad (2.3.13)$$

which for the FRW metric (neglecting curvature) becomes

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + V'(\varphi) = 0\,. \quad (2.3.14)$$

Note the appearance of the friction term $3H\dot{\varphi}$, which is due to the universe expansion.

Now, we split the inflaton field as:

$$\varphi(t,\mathbf{x}) = \varphi_0(t) + \delta\varphi(t,\mathbf{x}) \quad (2.3.15)$$

where $\varphi_0(t)$ is the classical (infinite wavelength) homogeneous field, and $\delta\varphi(t,\mathbf{x})$ is the quantum fluctuation around φ_0. Since quantum fluctuations are much smaller than the classical value, we can well neglect them in order to study the classical evolution and the expansion of the universe. To simplify the notation, in the following of this Section when we write φ we will mean φ_0.

The homogeneous classical field behaves like a perfect fluid with energy density and pressure given by

$$\rho_\varphi = T_{00} = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad (2.3.16)$$

$$P_\varphi = \frac{1}{3}T^i_{\ i} = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)\,. \quad (2.3.17)$$

Therefore, if

$$\dot{\varphi}^2 \ll V(\varphi) \quad (2.3.18)$$

we obtain the condition

$$P_\varphi \simeq -\rho_\varphi \quad (2.3.19)$$

from which we realize that a period of inflation (a quasi-de Sitter stage) takes place if the Universe is dominated by the energy density of a scalar field whose potential energy dominates over its kinetic energy.

Notice that ordinary matter fields and the spatial curvature K are usually neglected during inflation because their contribution to the energy density is exponentially redshifted away during the accelerated expansion; small inhomogeneities are wiped out too, thus justifying the use of the background FRW metric.

2.3.3 Slow-roll conditions

Now, let us better quantify the conditions under which a scalar field may give rise to a period of inflation. The condition (2.3.18) means that the scalar field is slowly rolling down its potential: this is the reason why such a period is called slow-roll. This can be achieved if the field is in a region where the potential is sufficiently flat, so we may also expect that $\ddot{\varphi}$ is negligible as well.

In the slow-roll approximation the FRW equation (2.3.4) becomes

$$H^2 \simeq \frac{8\pi G}{3} V(\varphi) \tag{2.3.20}$$

and the equation of motion (2.3.14) gives

$$3H\dot{\phi} = -V'(\varphi)\,. \tag{2.3.21}$$

It is useful to introduce the so-called *slow-roll* parameters:

$$\varepsilon \equiv \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2, \tag{2.3.22}$$

$$\eta \equiv M_P^2 \frac{V''}{V}, \tag{2.3.23}$$

and in terms of these we can say that equations (2.3.20) and (2.3.21) are valid if $\varepsilon \ll 1$, $|\eta| \ll 1$, which are called *slow-roll conditions*.

The slow-roll approximation is a sufficient condition for inflation. In fact, we must have

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0 \tag{2.3.24}$$

and so, if $\dot{H} < 0$, this becomes

$$-\frac{\dot{H}}{H^2} < 1\,. \tag{2.3.25}$$

If we substitute the slow-roll equations (2.3.20) and (2.3.21) we have

$$-\frac{\dot{H}}{H^2} \simeq \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 = \varepsilon \ll 1 \tag{2.3.26}$$

and so inflation is attained if the slow-roll conditions are satisfied.

2.3.4 Attractor behaviour

The success of the slow-roll approximation is due to the attractor behaviour of the slow-roll solutions: that is, solutions of the complete equations (2.3.14) and (2.3.4) from a wide range of initial conditions exponentially approach the slow-roll solutions.

To show this, we start from the equation

$$\dot{H} = \frac{dH}{d\varphi}\dot{\varphi} = -\frac{1}{2}\dot{\varphi}^2 \Longrightarrow \dot{\varphi} = -2\frac{dH}{d\varphi} \tag{2.3.27}$$

and so

$$3\frac{\frac{1}{2}\dot{\varphi}^2}{\frac{1}{2}\dot{\varphi}^2 + V(\varphi)} = 2\left(\frac{H'(\varphi)}{H(\varphi)}\right) \simeq \varepsilon(\varphi) \ll 1\,. \tag{2.3.28}$$

Now, suppose we have a slow-roll solution φ_0, $H(\varphi_0)$: if we consider a perturbation of it, $\varphi + \delta\varphi$, $H_0 + \delta H$, keeping only terms linear in the perturbations we find

$$H_0'\delta H' \simeq \frac{3}{2M_P^2}H_0\delta H \tag{2.3.29}$$

which has the solution

$$\delta H(\varphi) \simeq \delta H(\varphi_0)\exp\left[\frac{3}{2}\int_{\varphi_0}^{\varphi}\frac{H_0}{H_0'}d\tilde{\varphi}\right] \tag{2.3.30}$$

where the exponential is negative and big: so the slow roll solutions are attractive solutions.

2.3.5 Solution to the horizon problem

During inflation, the Hubble radius H^{-1} is nearly constant. So, any length scale that entered the horizon during the matter or radiation dominated periods, has been causally connected with any other at some primordial stage, because such scales are exponentially reduced. This can explain both the problem of the homogeneity of the CMB and the initial condition problem of small cosmological perturbations: in fact, once the physical lengths of interest are within the horizon, microphysics can act, the universe can be made approximately homogeneous and small primordial inhomogeneities can be created.

The solution of the horizon problem requires a lower bound on the duration of inflation.

A necessary condition to solve the horizon problem is that the largest scale we observe today, that is the present horizon H_0^{-1}, was reduced during inflation to a value $\lambda_{H_0}(t_i)$ smaller than H_I^{-1}. This condition gives

$$\lambda_{H_0}(t_i) = H_0^{-1}\frac{a_i}{a_0} = H_0^{-1}\frac{a_i}{a_f}\frac{a_f}{a_0} = H_0^{-1}e^{-N}\frac{T_0}{T_f} < H_I^{-1} \tag{2.3.31}$$

where T_f is the temperature at the end of inflation, which can be identified with the temperature at the beginning of the radiation dominated era, and N is the number of e-folds defined in eq. (2.3.10). Finally we get

$$\begin{aligned} N < \ln\left(\frac{T_0}{H_0}\right) - \ln\left(\frac{T_f}{H_I}\right) \simeq \ln\left(\frac{2.35\times 10^{-4}\text{eV}}{1.51\times 10^{-31}\text{eV}}\right) - \ln\left(\frac{T_f}{H_I}\right) \simeq \\ \simeq 62 - \ln\left(\frac{T_f}{H_I}\right)\,, \end{aligned} \tag{2.3.32}$$

which is the minimum number of e-folds of inflation required to solve the horizon problem.

2.3.6 Solution to the flatness problem

To show how inflation solves the flatness problem, we begin by noting that

$$\Omega - 1 = \frac{K}{a^2 H^2} \propto \frac{1}{a^2} \tag{2.3.33}$$

since during inflation $H \simeq$ const. Now, we have seen that to reproduce a value of $(\Omega_0 - 1)$ of order unity today, the initial value of $(\Omega - 1)$ at the Planck time must be $|\Omega - 1| \simeq 10^{-60}$. Now, since we identify the beginning of the radiation dominated era with the end of inflation, and the time scale of inflation is the Planck time, we must have $|\Omega - 1|_{t_f} \simeq 10^{-60}$. But this value comes naturally from the expoonential expansion: in fact, during inflation

$$\frac{|\Omega - 1|_{t_f}}{|\Omega - 1|_{t_i}} = \left(\frac{a_i}{a_f}\right)^2 = e^{-2N} \tag{2.3.34}$$

and if $|\Omega - 1|_{t_i}$ is of order unity and not finely tuned, it is enough to take $N \simeq 70$ to solve the flatness problem.

We can say that the flatness of our universe is a generic prediction of inflation, and this is confirmed by the current data on CMB anisotropies. However, we have to specify that inflation does not change the global geometric properties of spacetime; what it does is simply to magnify the curvature radius $R_{\rm curv} = \frac{H^{-1}}{\sqrt{|\Omega - 1|}}$, so that locally the universe appears flat with a great precision.

Chapter 3

Cosmological perturbations

3.1 Introduction

After having discussed the expansion of the Universe during the inflationary period, we now turn to the other key feature of inflation, namely the generation of the cosmological perturbations.

We start by studying how fluctuations in a scalar field are produced during an inflationary stage. Then, since fluctuations in the inflaton field are tightly coupled to perturbations in the metric, we briefly review the theory of cosmological perturbations up to second order, to be able to study the evolution of non-linearities that will be the main argument of this work.

Finally, we introduce the gauge-invariant curvature perturbation, a very useful quantity to characterize the perturbations, and we show it is conserved on large scales in the case of adiabatic perturbations.

3.2 Generation of perturbations

We now come to the very important issue of the generation of the primordial fluctuations which are the seeds for the cosmological perturbations [4].

In the Hot Big Bang model, these had to be put in by hand; we now see that inflation provides a natural mechanism for their generation, namely by the quantum fluctuations of some field, although there are many possible scenarios for the production of the perturbations.

However, as the mechanism of production and evolution of the quantum fluctuations is common to any scalar field, let us study the quantum fluctuations of a generic scalar field during a de Sitter stage.

3.2.1 Quantum fluctuations of a scalar field during a de Sitter stage

Let us consider the case of a scalar field $\chi(t,\mathbf{x})$ with an effective potential $V(\chi)$ in a de Sitter stage, during which $H = \text{const}$.

The equation of motion for the field is

$$\ddot{\chi} + 3H\dot{\chi} - \frac{1}{a^2}\nabla^2\chi + V'(\chi) = 0\,. \tag{3.2.1}$$

We split the scalar field in a classical homogeneous part and its fluctuations:

$$\chi(t,\mathbf{x}) = \chi(t) + \delta\chi(t,\mathbf{x}) \tag{3.2.2}$$

so, perturbing eq. (3.2.1) we obtain for the fluctuations

$$\ddot{\delta\chi} + 3H\dot{\delta\chi} - \frac{1}{a^2}\nabla^2\delta\chi + V''(\chi)\delta\chi = 0\,. \tag{3.2.3}$$

Let us give a heuristic explanation of the reason why we expect that such fluctuations are generated.

If we differentiate the equation for the classical field with respect to time, we obtain

$$\dddot{\chi_0} + 3H\ddot{\chi}_0 + V''\dot{\chi}_0 = 0\,, \tag{3.2.4}$$

In the limit $\mathbf{k}^2 \ll a^2$ we disregard the gradient term in eq. (3.2.3) and we see that $\delta\chi$ and $\dot{\chi}_0$ solve the same equation. They are indeed the same solution, because the Wronskian of the equation is $W(t) = W_0 \exp\left[-3\int^t H d\tilde{t}\right]$ and goes to zero for large t.

Therefore $\delta\chi$ and $\dot{\chi}_0$ have to be related to each other by a constant of proportionality depending only on $\mathbf{x}$:

$$\delta\chi = -\dot{\chi}_0\delta t(\mathbf{x})\,, \tag{3.2.5}$$

so the field $\chi(\mathbf{x},t)$ will be of the form

$$\chi(\mathbf{x},t) = \chi_0(\mathbf{x}, t - \delta t(\mathbf{x}))\,. \tag{3.2.6}$$

This equation says that the scalar field at a given time t does not acquire the same value in all the space. Rather, when the field is rolling down its potential, it acquires different values at different spatial points $\mathbf{x}$, so it is not homogeneous and fluctuations are present.

Now we can rewrite eq. (3.2.3) in conformal time τ as

$$\delta\chi'' + 2\mathcal{H}\delta\chi' - \nabla^2\delta\chi + a^2 m_\chi^2\delta\chi = 0\,, \tag{3.2.7}$$

where $m_\chi^2(\tau) \equiv \frac{d^2V(\chi)}{d\chi^2}$ is an effective time-dependent mass for the field.

We expand the scalar field in Fourier modes

$$\delta\chi(\tau,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\delta\chi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{3.2.8}$$

and we perform the field redefinition

$$\delta\chi_{\mathbf{k}} = \frac{\delta\sigma_{\mathbf{k}}}{a} \tag{3.2.9}$$

to obtain finally the equation

$$\delta\sigma''_{\mathbf{k}} + \left(k^2 + a^2 m_\chi^2 - \frac{a''}{a}\right)\delta\sigma_{\mathbf{k}} = 0\,, \tag{3.2.10}$$

which is the Klein-Gordon equation with a time-dependent mass term, and can be derived from the effective action

$$\delta S_{\mathbf{k}} = \int d\tau \left[\frac{1}{2}(\delta\sigma'_{\mathbf{k}})^2 - \frac{1}{2}\left(k^2 + a^2 m_\chi^2 - \frac{a''}{a}\right)\delta\sigma^2_{\mathbf{k}}\right]\,, \tag{3.2.11}$$

which is the canonical action for a simple harmonic oscillator.

Using the well-known techniques of quantum field theory, we can quantize the field writing it as

$$\delta\sigma_{\mathbf{k}} = u_k(\tau)a_{\mathbf{k}} + u_k^*(\tau)a^\dagger_{\mathbf{k}} \tag{3.2.12}$$

where we introduce the creation and annihilation operators, which satisfy the commutation relations

$$[a_{\mathbf{k}}, a_{\mathbf{q}}] = 0\,, \quad \left[a_{\mathbf{k}}, a^\dagger_{\mathbf{q}}\right] = \delta^{(3)}(\mathbf{k}-\mathbf{q}) \tag{3.2.13}$$

and the modes $u_k(\tau)$ are normalized as

$$u_k^* u_k' - u_k u_k^{*\prime} = -i \tag{3.2.14}$$

to satisfy the usual canonical commutation relations between $\delta\sigma$ and its conjugate momentum $\Pi = \delta\sigma'$.

The modes $u_k(\tau)$ obey the equation of motion

$$u_k'' + \left(k^2 + a^2 m_\chi^2 - \frac{a''}{a}\right)u_k = 0 \tag{3.2.15}$$

which has an exact solution in the case of a de Sitter stage; however, before recovering it, it is instructive to study its behaviour in the sub-horizon and super-horizon limits.

In a de Sitter stage $a = -\frac{1}{H\tau}$, so $a' = \frac{1}{H\tau^2}$ and $a'' = -\frac{2}{H\tau^3} = \frac{2}{\tau^2}a$; on subhorizon scales we have $k^2 \gg a^2H^2 \simeq \frac{a''}{a}$, so eq.(3.2.15) becomes

$$u_k'' + k^2 u_k = 0 \tag{3.2.16}$$

whose solution is

$$u_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau} \; (k \gg aH) \,. \tag{3.2.17}$$

Thus fluctuations with wavelength within the horizon oscillate as in flat space-time: this is what we expect, because in this limit the space-time can well be approximated as flat.

On superhorizon scales $k^2 \ll \frac{a''}{a}$ and eq.(3.2.15) becomes

$$u_k'' + \left(a^2 m_\chi^2 - \frac{a''}{a} \right) u_k = 0 \tag{3.2.18}$$

which is easy to solve for a massless field ($m_\chi^2 = 0$): there are two solutions, a growing and a decaying mode

$$u_k(\tau) = B_+(k) a + B_-(k) \frac{1}{a^2} \,. \tag{3.2.19}$$

We can fix the amplitude of the growing mode by matching this solution to the plane wave solution when the fluctuation with wavenumber k leaves the horizon ($k = aH$), finding

$$|B_+(k)| = \frac{1}{a\sqrt{2k}} = \frac{H}{\sqrt{2k^3}} \,, \tag{3.2.20}$$

so that the quantum fluctuations of the original field χ are constant:

$$|\delta\chi_{\mathbf{k}}| = \frac{|u_k|}{a} \simeq \frac{H}{\sqrt{2k^3}} \,. \tag{3.2.21}$$

Now we derive the exact solution to eq. (3.2.15), which in the case of a massless field reads

$$u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau} \right) \; (m_\chi^2 = 0) \tag{3.2.22}$$

with the initial condition $u_k(\tau) \simeq \frac{1}{\sqrt{2k}} e^{-ik\tau}$ for $k \gg aH$.

In a de Sitter stage we have

$$\frac{a''}{a} - m_\chi^2 a^2 = \frac{2}{\tau^2} \left(1 - \frac{1}{2} \frac{m_\chi^2}{H^2} \right) \tag{3.2.23}$$

so that we can recast eq. (3.2.15) in the form

$$u_k''(\tau) + \left(k^2 - \frac{\nu_\chi^2 - \frac{1}{4}}{\tau^2} \right) u_k(\tau) = 0 \tag{3.2.24}$$

where

$$\nu_\chi^2 = \frac{9}{4} - \frac{m_\chi^2}{H^2} \,. \tag{3.2.25}$$

When the mass is constant in time, eq. (3.2.24) is a Bessel equation whose general solution for real ν_χ, that is, for light fields such that $m_\chi < \frac{3}{2}H$, reads

$$u_k(\tau) = \sqrt{-\tau}\left[c_1(k)H^{(1)}_{\nu_\chi}(-k\tau) + c_2(k)H^{(2)}_{\nu_\chi}(-k\tau)\right], \qquad (3.2.26)$$

where $H^{(1)}_{\nu_\chi}$, $H^{(2)}_{\nu_\chi}$ are the Hankel functions of first and second kind, respectively.

As boundary condition, we impose that in the ultraviolet regime $k \gg aH$ ($-k\tau \gg 1$) the solution matches the plane-wave solution $u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}$ we expect in flat space-time. Knowing that

$$H^{(1)}_{\nu_\chi}(x \gg 1) \simeq \sqrt{\frac{2}{\pi x}}e^{i\left(x-\frac{\pi}{2}\nu_\chi-\frac{\pi}{4}\right)},\ H^{(2)}_{\nu_\chi}(x \gg 1) \simeq \sqrt{\frac{2}{\pi x}}e^{-i\left(x-\frac{\pi}{2}\nu_\chi-\frac{\pi}{4}\right)}, \qquad (3.2.27)$$

we then set $c_2(k) = 0$ and $c_1(k) = \frac{\sqrt{\pi}}{2}e^{i\left(\nu_\chi+\frac{1}{2}\right)\frac{\pi}{2}}$, which also satisfy the normalization condition (3.2.14).

So the exact solution becomes

$$u_k(\tau) = \frac{\sqrt{\pi}}{2}e^{i\left(\nu_\chi+\frac{1}{2}\right)\frac{\pi}{2}}\sqrt{-\tau}H^{(1)}_{\nu_\chi}(-k\tau)\,. \qquad (3.2.28)$$

On superhorizon scales, since

$$H^{(1)}_{\nu_\chi}(x \ll 1) \simeq \sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}2^{\nu_\chi-\frac{3}{2}}\frac{\Gamma(\nu_\chi)}{\Gamma(\frac{3}{2})}x^{-\nu_\chi} \qquad (3.2.29)$$

the solution (3.2.28) has the limiting behaviour

$$u_k(\tau) \simeq e^{i\left(\nu_\chi-\frac{1}{2}\right)\frac{\pi}{2}}2^{\nu_\chi-\frac{3}{2}}\frac{\Gamma(\nu_\chi)}{\Gamma(\frac{3}{2})}\frac{1}{\sqrt{2k}}(-k\tau)^{\frac{1}{2}-\nu_\chi}\,. \qquad (3.2.30)$$

Thus we find that on superhorizon scales the fluctuations of the scalar field $\delta\chi_k = \frac{u_k}{a}$ are not exactly constant, but acquire a tiny dependence upon time

$$|\delta\chi_{\mathbf{k}}| = 2^{\nu_\chi-\frac{3}{2}}\frac{\Gamma(\nu_\chi)}{\Gamma(\frac{3}{2})}\frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_\chi} \quad (k \ll aH). \qquad (3.2.31)$$

We introduce the parameter $\eta_\chi = \frac{m_\chi^2}{3H^2}$: if the field is very light $\frac{3}{2}-\nu_\chi \simeq \eta_\chi$, and to lowest order in η_χ we have

$$|\delta\chi_{\mathbf{k}}| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\eta_\chi} \quad (k \ll aH). \qquad (3.2.32)$$

This equation shows a crucial result: when the scalar field is light, its quantum fluctuations generated on subhorizon scales are gravitationally amplified and stretched to superhorizon scales because of the accelerated expansion.

3.2.2 Quantum fluctuations of a scalar field during a quasi-de Sitter stage

During inflation, the universe does not exactly evolve in a pure de Sitter expansion, with $a(\tau) = -\frac{1}{H\tau}$, $\dot{H} = 0$; rather we have what we call a quasi-de Sitter expansion, with H evolving slowly in time as $\dot{H} = -\varepsilon H^2$.

As we have just seen, fluctuations on superhorizon scales will be generated only if the scalar field has a small effective mass: so we consider $\eta_\chi = \frac{m_\chi^2}{3H^2} \ll 1$, and we can make an expansion to lowest order in η_χ and in the slow-roll parameter ε.

From the definition of conformal time $d\tau = \frac{dt}{a}$ we find

$$a(\tau) \simeq -\frac{1}{H}\frac{1}{\tau(1-\varepsilon)} \tag{3.2.33}$$

and

$$\frac{a''}{a} = a^2 H^2 \left(2 + \frac{\dot{H}}{H}\right) \simeq \frac{2}{\tau^2}\left(1 + \frac{3}{2}\varepsilon\right) . \tag{3.2.34}$$

This way we obtain again the Bessel equation (3.2.24), where now ν_χ is given by

$$\nu_\chi \simeq \frac{3}{2} + \varepsilon - \eta_\chi \, , \tag{3.2.35}$$

which we can treat as a constant since the time derivatives of the slow-roll parameters are $\dot{\varepsilon}, \dot{\eta} \simeq O(\varepsilon^2, \eta^2)$.

Thus the solution is given by eq. (3.2.28) with the new expression for η_χ; on large scales we find

$$|\delta\chi_k| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\eta\chi-\varepsilon} \simeq \frac{H}{\sqrt{2k^3}}\left[1 + (\eta_\chi - \varepsilon)\ln\left(\frac{k}{aH}\right)\right] , \tag{3.2.36}$$

and we obtain

$$|\dot{\delta\chi_k}| \simeq |H\eta_\chi \delta\chi_k| \ll |H\delta\chi_k| \tag{3.2.37}$$

which shows that the fluctuations are nearly frozen on superhorizon scales.

3.2.3 The power spectrum

A useful quantity to characterize the properties of the perturbations is the *power spectrum.* Given a stochastic field $f(t, \mathbf{x})$ which we can expand in a Fourier integral

$$f(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} f_{\mathbf{k}}(t) \tag{3.2.38}$$

its (dimensionless) power spectrum $\mathcal{P}_f(k)$ is defined by

$$\left\langle f_{\mathbf{k}_1} f^*_{\mathbf{k}_2} \right\rangle \equiv \frac{2\pi^2}{k^3}\mathcal{P}_f(k)\delta^{(3)}(\mathbf{k}_1 - \mathbf{k}_2) \tag{3.2.39}$$

where the angle brackets denote ensemble average.

The two-point correlation function is given by:

$$\langle f(t,\mathbf{x}_1)f(t,\mathbf{x}_2)\rangle = \int \frac{d^3k}{4\pi k^3}\mathcal{P}_f(k)e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)} = \int \frac{dk}{k}\frac{\sin(kx)}{kx}\mathcal{P}_f(k) \tag{3.2.40}$$

where $x = |\mathbf{x}_1 - \mathbf{x}_2|$.

It is standard practice to define the *spectral index* $n_f(k)$ through

$$n_f(k) - 1 \equiv \frac{d\ln \mathcal{P}_f(k)}{d\ln k} \tag{3.2.41}$$

since in many models of inflation the spectrum can well be approximated by a power law.

For the fluctuations of a scalar field we have:

$$\begin{aligned}\left\langle \delta\chi_{\mathbf{k}}\delta\chi^*_{\mathbf{q}}\right\rangle &= \frac{1}{a^2}\left\langle \delta\sigma_{\mathbf{k}}\delta\sigma^*_{\mathbf{q}}\right\rangle = \frac{1}{a^2}\left\langle (u_k a_{\mathbf{k}} + u^*_k a^\dagger_{\mathbf{k}})(u^*_q a^\dagger_{\mathbf{q}} + u_q a_{\mathbf{q}})\right\rangle = \\ &= \frac{1}{a^2}\left\langle (u_k a_{\mathbf{k}})(u^*_q a^\dagger_{\mathbf{q}})\right\rangle = \frac{1}{a^2}u_k u^*_q \left\langle \left[a_{\mathbf{k}}, a^\dagger_{\mathbf{q}}\right]\right\rangle = \frac{|u_k|^2}{a^2}\delta^{(3)}(\mathbf{k}-\mathbf{q})\end{aligned} \tag{3.2.42}$$

so, using the definition (3.2.39), we find

$$\mathcal{P}_{\delta\chi}(k) = \frac{k^3}{2\pi^2}|\delta\chi_k|^2\,. \tag{3.2.43}$$

In the case of a scalar field in a de Sitter stage, considering $m_\chi \ll \frac{3}{2}H$, from eq. (3.2.32) we compute the spectrum on superhorizon scales:

$$\mathcal{P}_{\delta\chi}(k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu_\chi}\,. \tag{3.2.44}$$

Therefore we have a nearly scale-invariant (or Harrison-Zel'dovich) spectrum: the spectral index is

$$n_{\delta\chi} - 1 = 3 - 2\nu_\chi = 2\eta_\chi \ll 1\,; \tag{3.2.45}$$

in particular for a massless field we have exactly scale-invariance on superhorizon scales.

For a scalar field in a quasi-de Sitter stage we use eq. (3.2.36) to find, on superhorizon scales,

$$\mathcal{P}_{\delta\chi}(k) \simeq \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{2\eta_\chi-\varepsilon}\,, \tag{3.2.46}$$

which could be also calculated using eq. (3.2.44) replacing H by H_k, i. e. the value at the time when the fluctuations with wavenumber k leave the horizon.

This gives the correct result because the fluctuations are nearly frozen on superhorizon scales.

3.2.4 Metric fluctuations during inflation

We have seen how perturbations of a generic scalar field are generated during a quasi-de Sitter expansion. The inflaton is a scalar field, so we will have quantum fluctuations of the inflaton field as well. However, the inflaton field is a special field, because by assumption it dominates the energy density of the universe during inflation. So, any perturbation of the inflaton field is a perturbation of the energy-momentum tensor:

$$\delta\varphi \Longrightarrow \delta T_{\mu\nu}\,. \tag{3.2.47}$$

Now, a perturbation in the energy momentum tensor implies, via the Einstein equations, a metric perturbation:

$$\delta T_{\mu\nu} \Longrightarrow \delta R_{\mu\nu} - \frac{1}{2}\delta\left(g_{\mu\nu}R\right) = 8\pi G\delta T_{\mu\nu} \Longrightarrow \delta g_{\mu\nu}\,. \tag{3.2.48}$$

On the other hand, a metric perturbation induces a backreaction on the evolution of the inflaton field through the perturbed Klein-Gordon equation:

$$\delta g_{\mu\nu} \Longrightarrow \delta\left[\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\right) + \frac{\partial V}{\partial\varphi}\right] \Longrightarrow \delta\varphi\,. \tag{3.2.49}$$

Therefore, we conclude that the perturbations of the inflaton field and that of the metric are tightly coupled to each other, and we have to study them together,

$$\delta\varphi \Longleftrightarrow \delta g_{\mu\nu}\,. \tag{3.2.50}$$

So, in order to compute the metric fluctuations produced during inflation, we now stop for a while to describe the basic tool we need, cosmological perturbation theory.

3.3 Metric perturbations

We know that the universe is quasi homogeneous and isotropic, so to describe its evolution on large scales we can neglect the perturbations and use a FRW model.

To describe inhomogeneities and anisotropies we now introduce cosmological perturbation theory (for a detailed account, see [5, 6, 7]). The main idea underlying perturbation theory is that we have a background FRW metric plus small perturbations; we then solve perturbatively the Einstein equations in these small perturbations.

Since observed anisotropies are of the order of 10^{-5}, one can think that studying perturbations at linear order is sufficient; however, as we are interested in computing the generation and evolution of non-Gaussian effects, we need a non linear analyisis, so we study perturbation theory up to second order.

We begin by writing the metric tensor as a background metric plus a perturbation:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta^{(1)} g_{\mu\nu} + \frac{1}{2}\delta^{(2)} g_{\mu\nu} \,. \tag{3.3.1}$$

Since we consider an homogeneous and isotropic background, we will take $g^{(0)}_{\mu\nu}$ as the more general FRW metric:

$$ds^{(0)2} = g^{(0)}_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)\left[-d\eta^2 + \gamma_{ij} dx^i dx^j\right] \tag{3.3.2}$$

where η is the conformal time (often denoted by τ in calculations during inflation), and γ_{ij} is the metric of a space of constant curvature K.

The components of a perturbed the FRW metric will have the form:

$$\delta g_{00} = -a^2(1+2\phi) \tag{3.3.3}$$

$$\delta g_{0i} = a^2 \hat{\omega}_i \tag{3.3.4}$$

$$\delta g_{ij} = a^2\left[(1-2\psi)\gamma_{ij} + \hat{\chi}_{ij}\right] \,, \tag{3.3.5}$$

and every component can be expanded up to second order as $\delta g = \delta^{(1)} g + \frac{1}{2}\delta^{(2)} g$.

We can make a similar splitting for the perturbed energy-momentum tensor:

$$T^\mu_\nu = T^{\mu(0)}_\nu + \delta^{(1)} T^\mu_\nu + \delta^{(2)} T^\mu_\nu \tag{3.3.6}$$

where the unperturbed part will describe a perfect fluid being of the form

$$T^{\mu(0)}_\nu = (\rho + P)\, u^\mu u_\nu + P\delta^\mu_\nu \tag{3.3.7}$$

where ρ is the energy density, P the pressure, and u^μ is the fluid four-velocity subject to the normalization $g^{(0)}_{\mu\nu} u^\mu u^\nu = -1$.

The energy density will be expanded as

$$\rho(\eta, x^i) = \rho_0(\eta) + \delta\rho(\eta, x^i) = \rho_0(\eta) + \delta^{(1)}\rho(\eta, x^i) + \frac{1}{2}\delta^{(2)}\rho(\eta, x^i) \tag{3.3.8}$$

and the pressure is given by $P + \delta P = w(\rho + \delta\rho)$.

The velocity field up to second order will be

$$u^\mu = \frac{1}{a}\left(\delta^\mu_0 + v^\mu_{(1)} + \frac{1}{2} v^\mu_{(2)}\right) \,. \tag{3.3.9}$$

The normalization condition gives

$$v^0_{(1)} = -\phi^{(1)} \tag{3.3.10}$$

$$v^0_{(2)} = -\phi^{(2)} + 3\left(\phi^{(1)}\right)^2 + \hat{\omega}_i v^i_{(1)} + v^{(1)}_i v^i_{(1)} \,. \tag{3.3.11}$$

Now, it is very useful to split the perturbations into scalar, vector and tensor modes, according to their properties under *spatial* rotations on hypersurfaces of constant time. The reason for this splitting is that in linear theory these different modes decouple, and at second order one can obtain decoupled equations for the second order modes as well, albeit with a source term given by different first-order modes; moreover, this splitting helps to understand the physical meaning of the various perturbations.

A vector quantity v^i can be decomposed as

$$v^i = v_S^{|i} + v_V^i \tag{3.3.12}$$

$$v^i_{V\ |i} = 0 \tag{3.3.13}$$

where v_S is a scalar quantity, a vertical bar denotes a covariant derivative with respect to γ_{ij}, and v_V^i is a transverse (divergenceless) vector.

Similarly, a symmetric second-rank tensor can be decomposed as:

$$t^{ij} = t_S^{|ij} + (t_V^{i|j} + t_V^{j|i}) + t_T^{ij} \tag{3.3.14}$$

$$t^i_{V\ |i} = 0 \tag{3.3.15}$$

$$t^{ij}_{T\ |ij} = 0\,, \quad \gamma_{ij} t_T^{ij} = 0 \tag{3.3.16}$$

where t_S is a scalar function, t_V^i is a transverse vector and t_T^{ij} is a traceless transverse tensor.

To obtain a set of ordinary differential equations, at least in linear theory, it is useful to employ a harmonic decomposition in eigenfunctions of the Laplace-Beltrami operator of γ_{ij}:

$$\nabla^2 Q^{(0)} = -\,k^2 Q^{(0)} \tag{3.3.17}$$

$$\nabla^2 Q_i^{(\pm1)} = -\,k^2 Q_i^{(\pm1)} \tag{3.3.18}$$

$$\nabla^2 Q_{ij}^{(\pm2)} = -\,k^2 Q_{ij}^{(\pm2)}\,. \tag{3.3.19}$$

In a flat spacetime, these are simply plane waves:

$$Q^{(0)} = e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.3.20}$$

$$Q_i^{(\pm1)} = \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.3.21}$$

$$Q_{ij}^{(\pm2)} = -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{3.3.22}$$

where $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ are unit vectors spanning the plane orthogonal to $\mathbf{k}$.

Since our spacetime is flat or almost flat, we will be interested only in the flat FRW model, with $\gamma_{ij} = \delta_{ij}$. Our splitting of the perturbations to δg_{0i} and to δg_{ij} will be

$$\hat{\omega}_i^{(n)} = \partial_i \omega^{(n)} + \omega_i^{(n)} \tag{3.3.23}$$

$$\hat{\chi}_{ij}^{(n)} = D_{ij}\chi^{(n)} + \left(\partial_i \chi_j^{(n)} + \partial_j \chi_i^{(n)}\right) + \chi_{ij}^{(n)} \tag{3.3.24}$$

where $n = 1, 2$, ω_i and χ_i are transverse vectors, χ_{ij} is a symmetric, transverse and traceless tensor and $D_{ij} \equiv \partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2$ is a traceless operator, to have $\delta g^i{}_i = -6a^2\psi$.

Finally, the perturbed line element will be written as

$$\begin{aligned} ds^2 = a^2 \big[&-(1+2\phi)d\tau^2 + (\partial_i\omega + \omega_i)d\tau dx^i + (1-2\psi)\delta_{ij} + \\ &+ \left(D_{ij}\chi + \partial_i\chi_j + \partial_j\chi_i + \chi_{ij}\right) dx^i dx^j \big] \end{aligned} \tag{3.3.25}$$

with every quantity expanded up to second order.

3.3.1 Gauge transformations

General relativity is a theory invariant under a change in the coordinates one uses to describe space-time: in other words, there is invariance under diffeomorphisms.

In perturbation theory we define a perturbation as the difference between the physical value of a quantity T and its background value T_0. So we are dealing with two different spacetimes, the real physical (perturbed) spacetime, and the background one, which in our case is a manifold endowed with a flat FRW metric, and our quantities T and T_0 are defined respectively on the physical and on the background manifold.

In order to make a comparison between the two quantities, we have to consider them at the same point, so we need to establish a one-to-one correspondence between the background and the perturbed manifold. Such a map is a gauge choice, and all possible choices are on the same footing because of the gauge-invariance of the theory: changing the map means making a gauge transformation, which will not alter our physical results [8, 9].

In perturbation theory, we consider a family of spacetime models, i.e. a family of manifolds endowed with a Lorentzian metric $\{(\mathcal{M}_\lambda, g_\lambda)\}$, where λ is a real parameter, and the metric satisfies the field equations $\mathcal{E}[g_\lambda] = 0$.

We will assume that g_λ depends smoothly on the parameter λ, so that λ itself is a measure of the difference between $(\mathcal{M}_\lambda, g_\lambda)$ and the background $(\mathcal{M}_0, g_0)$.

In some applications, λ is a dimensionless parameter naturally arising from the problem at hand; in our case, it is just a formal parameter and in the end, for convenience, one can choose $\lambda = 1$ for the physical spacetime.

In order to define the perturbations in a given tensor T, we must find a way to compare T_λ with T_0, and this requires a prescription for indentifying points of $\mathcal{M}_\lambda$ with those of $\mathcal{M}_0$. This is easily accomplished by assigning a diffeomorphism $\phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_\lambda$. The perturbation can now be defined simply as

$$\Delta T_\lambda \equiv \phi^*_\lambda T - T_0 \,, \tag{3.3.26}$$

where $\phi^*_\lambda T$ is the pull-back of T on $\mathcal{M}_0$.

Now, the first term on the r.h.s can be Taylor expanded in λ to get

$$\Delta T_\lambda = \sum_{r=1}^{+\infty} \frac{\lambda^r}{r!} \delta^r T \,, \tag{3.3.27}$$

where

$$\delta^r T = \left. \frac{d^r \phi_\lambda^* T}{d\lambda^r} \right|_{\lambda=0} \tag{3.3.28}$$

is the r-th order perturbation of T.

It is worth noticing that ΔT_λ and $\delta^r T$ are defined on $\mathcal{M}_0$: this formalizes the stament one commonly finds, that "perturbations are fields living in the background".

It is important to appreciate that the parameter λ is used also to perform the expansion (3.3.27), and determines what is meant by perturbations of the r-th order.

Now, consider two different diffeomorphisms, ϕ_λ and ψ_λ, carrying the points of $\mathcal{M}_0$ into those of $\mathcal{M}_\lambda$. Suppose that coordinates x^μ have been defined on $\mathcal{M}_0$ so that the point $p \in \mathcal{M}_0$ has coordinates $x^\mu(p)$, and let us use the map ψ_λ: now $O = \psi_\lambda(p)$ is the point in $\mathcal{M}_\lambda$ corresponding to p, to which ψ_λ assigns the same coordinate labels.

If we perform a gauge transformation, choosing the map ϕ_λ, we can think of O as the point of $\mathcal{M}_\lambda$ corresponding to a different point q of $\mathcal{M}_0$, with coordinates $\tilde{x}^\mu$: then $O = \psi_\lambda(p) = \phi_\lambda(q)$.

Thus the change of the correspondence between points of $\mathcal{M}_0$ and of $\mathcal{M}$ (the gauge transformation) may equally well be described by a one-to-one correspondence between points in the background.

In fact, we start from p, we carry it over to $O = \psi_\lambda(p)$ on $\mathcal{M}_\lambda$ and we come back to q on $\mathcal{M}_0$ with $q = \phi_\lambda^{-1}(O)$; the net result is a transformation $\Phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0$ defined as $\Phi_\lambda(p) \equiv \phi_\lambda^{-1}(\psi_\lambda(p)) = q$. In terms of coordinates, we have that the coordinates of q, $\tilde{x}^\mu(q, \lambda) = \Phi_\lambda^\mu(x^\alpha(p))$ are one parameter functions of those of p. Such a transformation, that in a fixed coordinate system moves each point to another, is often called an active coordinate transformation. This is opposed to a passive coordinate transformation, which change coordinate labels at each point.

Now consider a tensor field T_λ defined on each $\mathcal{M}_\lambda$. With ψ_λ and ϕ_λ we can define in two different ways a representation on $\mathcal{M}_0$, which we will denote as $T(\lambda)$ and $\tilde{T}(\lambda)$. These are defined on $\mathcal{M}_0$, and their components are related by Φ_λ.

Since we can compare these fields with T_0, we can define the perturbations as

$$\Delta T(\lambda) \equiv T(\lambda) - T_0 \tag{3.3.29}$$

$$\Delta \tilde{T}(\lambda) \equiv \tilde{T}(\lambda) - T_0 \,. \tag{3.3.30}$$

This ambiguity of definition is the gauge dependence of the perturbations.

Up to second order in the perturbations, an infinitesimal coordinate transformation is represented as

$$\tilde{x}^\mu = x^\mu + \xi^\mu_{(1)} + \frac{1}{2}\left(\xi^\mu_{(1)\,,\nu}\xi^\nu_{(1)} + \xi^\mu_{(2)}\right) \tag{3.3.31}$$

where $\xi^\mu_{(r)}(x)$ are independent vector fields defining the gauge transformation, to be regarded as quantities of the same order as the perturbation variables.

As for the perturbation variables, we can split the components of $\xi^\mu(r)$ into scalar and vector parts:

$$\xi^0_{(r)} = \alpha_{(r)} \tag{3.3.32}$$

$$\xi^i_{(r)} = \partial^i\beta_{(r)} + d^i_{(r)} \tag{3.3.33}$$

with $d^i_{(r)}$ a transverse vector, i. e. $\partial_i d^i_{(r)} = 0$.

From a practical point of view, we have seen that fixing a gauge is equivalent to fix a coordinate system, or in other words to choose a slicing of spacetime into hypersurfaces at constant τ and a threading of spacetime into lines at constant x^i. In our case, the function $\xi^0_{(r)}$ selects constant τ hypersurfaces, while $\xi^i_{(r)}$ selects the spatial coordinates within those hypersurfaces.

The transformation of a tensor $T = T_0 + \delta^{(1)}T + \frac{1}{2}\delta^{(2)}T$ corresponding to the coordinate transformation (3.3.31) is

$$\delta^{(1)}\tilde{T} = \delta^{(1)}T + \pounds_{\xi_{(1)}}T_0 \tag{3.3.34}$$

$$\delta^{(2)}\tilde{T} = \delta^{(2}T + 2\pounds_{\xi_{(1)}}\delta^{(1)}T + \pounds^2_{\xi_{(1)}}T_0 + \pounds_{\xi_{(2)}}T_0 \tag{3.3.35}$$

where $\pounds_{\xi_{(r)}}$ is the Lie derivative along the vector $\xi_{(r)}$.

We recall that for any given tensor $T^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m}$ the Lie derivative along ξ^μ is defined as

$$\pounds_\xi T^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} = \xi^\lambda\partial_\lambda T^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m} - \sum_{i=1}^{n} T^{\mu_1\ldots\lambda\ldots\mu_n}{}_{\nu_1\ldots\nu_m}\partial_\lambda\xi^{\mu_i} + \sum_{i=1}^{m} T^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\lambda\ldots\nu_m}\partial_{\nu_i}\xi^\lambda \tag{3.3.36}$$

where ∂_μ can be replaced by any derivative operator.

For example, the perturbations in the metric tensor up to first order transform as

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \xi^\lambda\partial_\lambda g^{(0)}_{\mu\nu} + g^{(0)}_{\lambda\nu}\partial_\mu\xi^\lambda + g^{(0)}_{\mu\lambda}\partial_\nu\xi^\lambda \tag{3.3.37}$$

where $g^{(0)}_{\mu\nu}$ is the background metric.

Explicitly, for the perturbation variables defined by eqs. (3.3.3-3.3.5), we will have:

$$\tilde{\phi}_{(1)} = \phi_{(1)} + \alpha'_{(1)} + \mathcal{H}\alpha_{(1)} \tag{3.3.38}$$

$$\tilde{\omega}^{(1)}_i = \omega^{(1)}_i - \partial_i\alpha^{(1)} + \partial_i\beta^{(1)\prime} + d^{(1)\prime}_i \tag{3.3.39}$$

$$\tilde{\psi}_{(1)} = \psi_{(1)} - \frac{1}{3}\nabla^2\beta_{(1)} - \mathcal{H}\alpha_{(1)} \tag{3.3.40}$$

$$\tilde{\chi}^{(1)}_{ij} = \chi^{(1)}_{ij} + 2D_{ij}\beta_{(1)} + \partial_i d^{(1)}_j + \partial_j d^{(1)}_i \tag{3.3.41}$$

and for the perturbations in the matter variables we will have:

$$\delta\tilde{\rho} = \delta\rho + \rho'_{(0)}\alpha_{(1)} \tag{3.3.42}$$

$$\tilde{v}^0_{(1)} = v^0_{(1)} - \mathcal{H}\alpha_{(1)} - \alpha'_{(1)} \tag{3.3.43}$$

$$\tilde{v}^i_{(1)} = v^i_{(1)} - \partial^i\beta'_{(1)} - d^{i\prime}_{(1)} \tag{3.3.44}$$

A quantity is defined to be gauge-invariant if it is not affected by a gauge transformation. In perturbation theory there are two approches one can follow: the first is to identify combinations of perturbations that are gauge-invariant quantities; the second is to choose a given gauge and work in that gauge. In the latter approach one has to define completely the gauge: otherwise, one has unphysical gauge modes propagating and the interpretation of the results is problematic.

We will find convenient to introduce gauge-invariant variables which we will use to follow the evolution of the perturbations from an early period of inflation till the epoch of radiation, matter and dark enegy domination. In particular, we will study the gauge-invariant definitions of the curvature perturbation.

3.3.2 Adiabatic and isocurvature perturbations

Arbitrary cosmological perturbations can be decomposed into *adiabatic* and *isocurvature* perturbations.

Adiabatic or *curvature* perturbations produce a net perturbation in the total energy density and in the intrinsic spatial curvature.

The perturbation in any scalar quantity X can be described by a unique perturbation in expansion with respect to the background, so

$$H\delta t = H\frac{\delta X}{\dot{X}}\,. \tag{3.3.45}$$

In particular, if we consider energy density and pressure,

$$\frac{\delta\rho}{\dot{\rho}} = \frac{\delta P}{\dot{P}} \tag{3.3.46}$$

which implies that $P = P(\rho)$ is a function of the energy density only: this explains why they are called adiabatic perturbations.

Isocurvature or *entropy* perturbations, instead, do not perturb the spatial curvature or the total energy density: rather, they are perturbations in the local equation of state.

One useful way of defining this type of perturbations is to give its value on uniform energy density hypersurfaces:

$$H \frac{\delta X}{\dot{X}}\bigg|_{\delta\rho=0} = H\left(\frac{\delta X}{\dot{X}} - \frac{\delta\rho}{\dot{\rho}}\right), \tag{3.3.47}$$

a quantity that vanishes for adiabatic perturbations.

If we have a set of fluids characterized by ρ_i, it is conventional to introduce the gauge invariant variables

$$S_{ij} = 3H\left(\frac{\delta\rho_i}{\dot{\rho}_i} - \frac{\delta\rho_j}{\dot{\rho}_j}\right) = 3\left(\zeta_i - \zeta_j\right). \tag{3.3.48}$$

which have the meaning of relative entropies.

One simple example of isocurvature perturbations is the baryon-to-photon ratio

$$S = \delta\frac{n_B}{n_\gamma} = \frac{\delta n_B}{n_B} - \frac{\delta n_\gamma}{n_\gamma}. \tag{3.3.49}$$

3.4 The gauge-invariant curvature perturbation

At linear order, the intrinsic spatial curvature on hypersurfaces of constant conformal time and for a flat universe is

$${}^{(3)}R = \frac{4}{a^2}\nabla^2\hat{\psi}^{(1)} \tag{3.4.1}$$

where for simplicity of notation we write

$$\hat{\psi}^{(1)} = \psi^{(1)} + \frac{1}{6}\nabla^2\chi^{(1)}. \tag{3.4.2}$$

For this reason, the quantity $\hat{\psi}^{(1)}$ is usually referred to as the *curvature perturbation*. However, the curvature perturbation $\hat{\psi}^{(1)}$ is not gauge-invariant, but it is defined only on a given slicing.

If we perform a coordinate transformation $x^\mu \to x^\mu - \xi^\mu_{(1)}$, from the formulae (3.3.38-3.3.41) we have

$$\begin{aligned}&\hat{\psi}^{(1)} = \psi^{(1)} + \frac{1}{6}\nabla^2\chi^{(1)} \to \\ &\to \psi^{(1)} - \frac{1}{3}\nabla^2\beta_{(1)} - \mathcal{H}\alpha_{(1)} + \frac{1}{6}\nabla^2\left(\chi^{(1)} + 2\beta^{(1)}\right) = \hat{\psi}^{(1)} - \mathcal{H}\alpha^{(1)}\end{aligned} \tag{3.4.3}$$

which is clearly not invariant under a change in the time slicing.

If we consider the *slicing of uniform energy density*, which is defined to be the slicing where the perturbation in the energy density vanishes, $\delta\rho = 0$, from eq. (3.3.42) we have $\alpha_{(1)} = \frac{\delta^{(1)}\rho}{\rho_0'}$. So we can define the *curvature perturbation on slices of unifom energy density* $\zeta^{(1)}$ [10]:

$$-\zeta^{(1)} \equiv \hat{\psi}^{(1)}\Big|_{\delta\rho=0} = \hat{\psi}^{(1)} + \mathcal{H}\frac{\delta^{(1)}\rho}{\rho_0'}\,. \tag{3.4.4}$$

This quantity is clearly gauge-invariant up to first order and it is an example of how to find a gauge-invariant quantity by selecting in an unambiguous way a proper time slicing. Notice that this quantity can also be regarded as the energy density perturbation on uniform curvature slices, where $\psi^{(1)} = \chi^{(1)} = 0$, the so-called *spatially flat gauge*. This is at the basis of a powerful formalism to compute the curvature perturbation, which we will describe in the following.

Up to second order, we can write $\zeta = \zeta^{(1)} + \frac{1}{2}\zeta^{(2)}$: we find [11, 12]

$$\begin{aligned}-\zeta^{(2)} = \tilde{\hat{\psi}}^{(2)}\Big|_{\delta\rho=0} =& \\ =&\hat{\psi}^{(2)} + \mathcal{H}\frac{\delta^{(2)}\rho}{\rho'} - 2\mathcal{H}\frac{\delta^{(1)}\rho'}{\rho'}\frac{\delta^{(1)}\rho}{\rho'} - 2\frac{\delta^{(1)}\rho}{\rho'}\left(\hat{\psi}^{(1)\prime} + 2\mathcal{H}\hat{\psi}^{(1)}\right) + \\ &+ \left(\frac{\delta^{(1)}\rho}{\rho'}\right)^2\left(\mathcal{H}\frac{\rho''}{\rho} - \mathcal{H}' - 2\mathcal{H}^2\right)\,.\end{aligned} \tag{3.4.5}$$

which is gauge-invariant with respect to the time slicing at second-order, being constructed over spatial hypersurfaces where $\delta^{(1)}\rho = \delta^{(2)}\rho = 0$.

The reason why it is useful to introduce the quantities in eqs. (3.4.4), (3.4.5) is that they are conserved on superhorizon scales when possible isocurvature perurbations are not present. This important feature allows us to follow the evolution of non linearities from an early period of inflation, until the epochs of radiation and subsequently matter and dark energy domination.

Especially during inflation, it is often useful to consider another gauge-invariant curvature perturbation, the so-called *comoving curvature perturbation*, usually denoted by $\mathcal{R}$. It is the curvature perturbation defined on comoving hypersurfaces, that is hypersufaces orthogonal to the world lines of comoving observers (observers who measure zero net momentum flow).

Its expression at first order, during single-field models of inflation, is

$$\mathcal{R}^{(1)} = \hat{\psi}^{(1)} + \frac{\mathcal{H}}{\varphi'}\delta^{(1)}\varphi\,. \tag{3.4.6}$$

On large scales the relation between $\mathcal{R}$ and ζ is $\mathcal{R}^{(1)} = -\zeta^{(1)}$.

3.5 Spectrum of the curvature perturbation

Now we are able to compute the curvature perturbation produced during inflation from the fluctuations of the inflaton field. We will do the calculation in the simple case of slow-roll, single field models of inflation, which we will call *standard scenario* for the generation of perturbations.

The evolution equation for the field is the Klein-Gordon equation $\Box\varphi = \frac{\partial V}{\partial\varphi}$. Perturbing it at linear order gives

$$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \nabla^2\delta\varphi + a^2\,\delta\varphi\,\frac{\partial^2 V}{\partial\varphi^2} + 2\phi\frac{\partial V}{\partial\varphi} - \varphi_0'\left(\phi' + 3\psi' + \nabla^2\omega\right) = 0 \quad (3.5.1)$$

where all the perturbations are first-order.

A straigthforward way to calculate the curvature perturbation is to solve this equation in the spatially flat gauge where $\psi = \chi = 0$. In this gauge, the perturbations of the scalar field correspond to the so-called Sasaki-Mukhanov gauge-invariant variable [13, 14]:

$$Q_\varphi = \delta^{(1)}\varphi + \frac{\varphi_0'}{\mathcal{H}}\hat{\psi}^{(1)}\,. \quad (3.5.2)$$

In terms of the field $\tilde{Q}_\varphi = aQ_\varphi$ and employing the perturbed Einstein equations, eq. (3.5.1) becomes

$$\tilde{Q}_\varphi'' + \left(k^2 - \frac{a''}{a} + a^2\mathcal{M}_\varphi^2\right)\tilde{Q}_\varphi = 0 \quad (3.5.3)$$

where

$$\mathcal{M}_\varphi^2 = \frac{\partial^2 V}{\partial\varphi^2} - \frac{8\pi G}{a^4}\frac{\partial}{\partial\tau}\left(\frac{a^2}{\mathcal{H}}\varphi'\right) \quad (3.5.4)$$

is an effective mass for the fluctuations in this gauge. To lowest order in the slow-roll parameters $\varepsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)$ and $\eta = \frac{1}{8\pi G}\frac{V''}{V}$ it is given by

$$\frac{\mathcal{M}_\varphi^2}{H^2} = 3\eta - 6\varepsilon\,. \quad (3.5.5)$$

Now we notice that the evolution equation for $\tilde{Q}_\varphi$ has the same form as eq. (3.2.15), and we can recast it in the form of a Bessel equation for the eigenvalues $u_k(\tau)$:

$$u_k'' + \left(k^2 - \frac{\nu_\varphi - \frac{1}{4}}{\tau^2}\right)u_k = 0 \quad (3.5.6)$$

with

$$\nu_\varphi \simeq \frac{3}{2} + 3\varepsilon - \eta\,. \quad (3.5.7)$$

So from our previous results we conclude that on superhorizon scales and to lowest order in the slow-roll parameters the fluctuations in Q_φ are

$$|Q_\varphi(k)| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_\varphi} . \qquad (3.5.8)$$

Now we consider the curvature perturbation on comoving hypersurfaces defined in eq. (3.4.6), and by comparison with the definition of Q in eq. (3.5.2) we find

$$\mathcal{R}^{(1)} = \frac{\mathcal{H}}{\varphi'} Q_\varphi . \qquad (3.5.9)$$

Finally, the power spectrum of the curvature perturbation on large scales reads

$$\mathcal{P}_\mathcal{R} = \left(\frac{H^2}{2\pi\dot\varphi}\right)^2 \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_\varphi} \simeq \left(\frac{H^2}{2\pi\dot\varphi}\right)^2_* \qquad (3.5.10)$$

where the asterisk means the quantity in brackets is to be evaluated at the epoch a given perturbation leaves the horizon.

Therefore we have the prediction for the spectral index of the curvature perturbation:

$$n_\mathcal{R} - 1 \equiv \frac{d\ln\mathcal{P}_\mathcal{R}}{d\ln k} = 3 - 2\nu_\phi = -6\varepsilon + 2\eta , \qquad (3.5.11)$$

and we see that inflation produces a nearly scale-invariant (also called Harrison-Zel'dovich) spectrum, which was thought to be responsible for the observed inhomogeneities in the distribution of galaxies even before inflation.

3.6 δN formalism

Now we want to describe a very powerful formalism to compute the curvature perturbation generated from inflation, and to show in a general way that it is conserved on large scales [15, 16].

Consider a foliation of spacetime $\{\Sigma(t)\}$, with t the time coordinate labeling the hypersurfaces, and let n^μ be the unit vector field normal to $\Sigma(t)$.

We can introduce the quantity $\vartheta = \nabla_\mu n^\mu$, which is the volume expansion rate of the hypersurfaces along the integral curve $\gamma(\tau)$ of n^μ. This volume expansion rate can be regarded as a sort of local defined Hubble parameter: in fact, for a FRW metric $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$ we have $n^\mu = (1,0,0,0)$ and

$$\frac{1}{3}\vartheta = \frac{1}{3}\nabla_\mu n^\mu = \frac{1}{3}\Gamma^i_{0i} = \frac{1}{3}3H = H . \qquad (3.6.1)$$

We can define a local number of e-foldings for each integral curve:

$$\mathcal{N} = \int_{\gamma(\tau)} \frac{1}{3}\vartheta d\tau \tag{3.6.2}$$

where τ is the proper time along the curve.

Now, let us calculate ϑ for the linearly perturbed FRW metric. We have $n^\mu = \frac{1}{a}(1-2\phi, -\omega^i)$, and we get

$$\begin{aligned}
\frac{1}{3}\vartheta =& \frac{1}{3}\left\{\partial_0\left[\frac{1}{a}(1-2\phi)\right] + \frac{1}{a}\partial_i(-\omega^i) + \Gamma^{(1)\mu}_{\mu 0}(1-2\phi) + \Gamma^{(1)\mu}_{\mu i}(-\omega^i)\right\} = \\
=& \frac{1}{3}\left[-\frac{a'}{a^2}(1-2\phi) - \frac{2}{a}\phi' - \frac{1}{a}\partial_i\partial^i\omega + \frac{1}{a}(4\mathcal{H} + \phi' - 3\psi')(1-2\phi)\right] = \\
=& \frac{1}{3}\left[-H + 2H\phi - 2\dot{\phi} - \frac{1}{a}\partial_i\partial^i\omega + 4H - \dot{\phi} - 3\dot{\psi} - 8H\phi\right] = \\
=& H\left[1 - \phi - \frac{1}{H}\dot{\psi} - \frac{1}{3H}\frac{1}{a^2}\partial_i\partial^i\omega\right] .
\end{aligned} \tag{3.6.3}$$

The last term is negligible on superhorizon scales so we can drop it, obtaining

$$\frac{1}{3}\vartheta \simeq H\left(1 - \phi - \frac{1}{H}\dot{\psi}\right) . \tag{3.6.4}$$

For the proper time we have $\frac{d\tau}{dt} = 1 + \phi$; therefore

$$\mathcal{N} = \int_{\gamma(\tau)} H\left(1 - \phi - \frac{1}{H}\dot{\psi}\right)(1+\phi)dt = \int_{\gamma(\tau)} (H - \dot{\psi})dt , \tag{3.6.5}$$

from which

$$\delta N \equiv \mathcal{N} - N = -\Delta\psi . \tag{3.6.6}$$

In particular, if we choose a foliation such that the initial hypersurface is flat and the final is comoving, we get the general formula for the comoving curvature perturbation

$$\delta N(\Sigma_f(t_1), \Sigma_c(t_2); \gamma(\tau)) = \mathcal{R}(t_2) . \tag{3.6.7}$$

Now, we take t_1 to be some time during inflation soon after horizon crossing, and t_2 to be some time after complete reheating, when $\mathcal{R}_c$ has become a constant (isocurvature perturbations, if any, are no longer present). Then $\delta\mathcal{N}$ can be regarded as a function of t_2 and of the field configuration $\varphi(t_1, x^i)$ on $\Sigma_f(t_1)$, using the slow-roll approximation to eliminate the dependence on $\dot{\varphi}(t_1)$. Therefore

$$\mathcal{R}_c(t_2, x_2^i) = \delta N = \frac{\partial N}{\partial \varphi^a}\delta\varphi^a_f(t_1, x_1^i) \tag{3.6.8}$$

is a useful and simple formula to compute the comoving curvature perturbation and so the spectrum of scalar perturbations.

Carrying out the derivative of the number of e-folds:

$$\mathcal{R}_c^{(1)} = \frac{\partial N}{\partial \tau}\frac{\partial \tau}{\partial \varphi^a}\delta\varphi_f^a(t_1, x_1^i) = \frac{\mathcal{H}}{\varphi^{a'}}\delta\varphi_f^a(t_1, x_1^i) \tag{3.6.9}$$

from which we reproduce our previous result (3.5.10), recalling that (in one-single field models) $\delta\varphi_f(t_1, x_1^i)$ satisfies (3.5.3).

This result can be easily extended to second (or higher) order:

$$\mathcal{R}_c^{(2)} = \frac{\partial N}{\partial \varphi^a}\delta\varphi_f^a(t_1, x_1^i) + \frac{1}{2}\frac{\partial^2 N}{\partial \varphi^a \partial_\varphi^b}\delta\varphi_f^a(t_1, x_1^i)\delta\varphi_f^b(t_1, x_1^i)\,. \tag{3.6.10}$$

3.7 Conservation of the curvature perturbation

3.7.1 First order conservation

We now show the most important feature of the gauge-invariant curvature perturbation: that is, its conservation on large scales if the perturbations are adiabatic.

Thanks to this conservation, it is relatively simple to follow the evolution of the perturbations since their production during inflation till the epoch of recombination, when they manifest themselves as large-scale CMB anisotropies.

It is possible to give a general proof of the conservation of the curvature perturbation, at any order in standard perturbation theory, using the gradient expansion tecnique: it follows solely from the local energy-momentum conservation, $u^\nu \nabla_\mu T^\mu_\nu = 0$, so being in principle applicable to a large class of alternative theories of gravity.

To better understand how this works, we present the calculation for the first order perturbation $\zeta^{(1)}$, and subsequently we will see the general proof.

Consider scalar perturbations to the FRW metric (eq.3.3.25) up to first order: for simplicity, in what follows we will indicate linear perturbations without a $^{(1)}$ superscript.

The costant time hypersurfaces are orthogonal to the unit time-like vector field

$$n^\mu = (1-\phi, -\frac{1}{2}\partial^i\omega)\,. \tag{3.7.1}$$

The expansion of the spatial hypersurfaces with respect to the proper time $d\tau = (1+\phi)dt$, along the worldlines of observers with 4-velocity n^μ is given by

$$\vartheta \equiv \nabla_\mu n^\mu = 3H(1-\phi) - 3\dot{\hat{\psi}} + \nabla^2\sigma \tag{3.7.2}$$

where σ, the scalar describing the shear in the sense that $\sigma_{ij} = (\partial_i\partial_j - \frac{1}{3}\delta_{ij})\sigma$, is

$$\sigma = \frac{1}{2}(\dot{\chi} - \omega)\,. \tag{3.7.3}$$

With respect to the coordinate time, the expansion is

$$\tilde{\vartheta} = (1+\phi)\vartheta = 3H - 3\dot{\hat{\psi}} + \nabla^2\sigma\,, \tag{3.7.4}$$

which can be rewritten as an evolution equation for $\hat{\psi}$ in terms of the perturbed expansion $\delta\tilde{\vartheta} \equiv \tilde{\vartheta} - 3H$:

$$\dot{\hat{\psi}} = -\frac{1}{3}\delta\tilde{\vartheta} + \frac{1}{3}\nabla^2\sigma\,. \tag{3.7.5}$$

Note that all this is independent of the field equation, being simply differential geometry.

Now we use the local conservation of energy-momentum along the normal n^μ, $n^\nu\nabla_\mu T^\mu_\nu = 0$, where $T^\mu_\nu = T^\mu_{(0)\nu} + \delta^{(1)}T^\mu_\nu$; it gives

$$\dot{\delta\rho} = -3H(\delta\rho + \delta P) + (\rho + P)\left[3\dot{\hat{\psi}} - \nabla^2\left(\sigma + v + \frac{1}{2}\omega\right)\right]\,, \tag{3.7.6}$$

where $\partial_i v$ is the scalar part of the velocity perturbation of the fluid.

In the uniform density gauge, $\delta\rho = 0$ and $-\zeta = \hat{\psi}$ (eq. 3.4.4), so the energy conservation equation (3.7.6) gives

$$\dot{\zeta} = -\frac{H}{\rho + P}\delta P_{\text{nad}} - \frac{1}{3}\nabla^2\left(\sigma + v + \frac{1}{2}\omega\right) \tag{3.7.7}$$

where $\delta P|_{\delta\rho=0} = \delta P_{\text{nad}}$.

So we can have ζ constant if two conditions are satisfied:

- there is no non-adiabatic pressure perturbations;
- the divergence of the 3-momentum on zero-shear hypersurfaces ($\sigma = \omega = 0$), is negligible.

In any case, on sufficiently large scales, gradient terms can be neglected, so we are left with

$$\dot{\zeta} = -\frac{H}{\rho + P}\delta P_{\text{nad}} \tag{3.7.8}$$

which implies the constancy of ζ if the pressure perturbation is adiabatic.

For a multi-fluid system we can define uniform-density hypersurfaces for each fluid and a quantity $\zeta_i \equiv -\hat{\psi} - \delta\rho_{(i)}/\dot{\rho}$. We see that ζ_i remains constant for adiabatic perturbations in any fluid whose T^μ_ν is locally conserved.

So, in the simple case of radiation and (non-interacting) CDM, both ζ_m and ζ_γ remain constant on super-horizon scales.

The total curvature perturbation (on uniform *total* density hypersurfaces) is given by

$$\zeta = \frac{\frac{4}{3}\rho_\gamma\zeta_\gamma + \rho_m\zeta_m}{\frac{4}{3}\rho_\gamma + \rho_m} \,. \tag{3.7.9}$$

In the radiation dominated era, $\rho_\gamma \gg \rho_m$, we have $\zeta_{\text{in}} \simeq \zeta_\gamma$, while during the subsequent matter dominated era $\zeta_{\text{fin}} \simeq \zeta_m$.

Since for adiabatic perturbations on large scales ζ has remained constant, we have the so-called *adiabaticity condition*

$$\zeta_\gamma = \zeta_m \,, \tag{3.7.10}$$

which employing the continuity equation $\rho' = -3\mathcal{H}\rho(1+w)$ can be written in the more usual form

$$\frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} = \frac{1}{3}\frac{\delta\rho_m}{\rho_m} \,. \tag{3.7.11}$$

3.7.2 General proof

Now we are going to define a non linear generalization of the usual curvature perturbation, and study its behaviour on large scales [17].

In this proof we make use of the gradient expansion method, which is an expansion in the spatial gradients of the inhomogeneities.

We consider a fixed time hypersurface, and multiply every spatial gradient ∂_i by a formal small parameter ε; then we keep only terms up to first order in ε, and then put $\varepsilon = 1$.

In common perturbation theory the formal parameter ε would be multiplying the perturbations themselves, as we have discussed previously, and Einstein equations are solved iteratively in powers of ε.

In this approach, however, one assumes that every quantity is smooth on some sufficiently large scale with comoving size k^{-1}: since in the unperturbed universe we have a natural scale, the Hubble radius H, we see that the parameter ε can be given a physical meaning, because it corresponds to

$$\varepsilon \equiv \frac{k}{aH} \,. \tag{3.7.12}$$

Therefore, the typical value of the gradient of a quantity f in Hubble units is εf, and at a fixed time the limit $\varepsilon \to 0$ corresponds to $k \to 0$.

The key physical assumption behind this method is that in the limit $\varepsilon \to 0$, corresponding to a sufficiently large smoothing scale, the universe becomes *locally* homogeneous and isotropic, where by locally we mean that a region larger than the Hubble radius, but smaller than the smoothing scale, is to be considered.

This assumption looks very reasonable, because otherwise we would not be allowed to consider an homogeneous background, as we do in standard perturbation theory: there are subleties involved with the implementation of a smoothing procedure, but this issue is beyon the scope of this work.

Metric It is useful to consider the (3+1) splitting of the metric (ADM formalism); our line element will be

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{3.7.13}$$

where N is the lapse function, N^i the shift vector, and γ_{ij} the spatial 3-metric.

The unit timelike vector normal to the constant time hypersurfaces is

$$n_\mu = (-N, 0)\,, \qquad n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right). \tag{3.7.14}$$

We write the 3-metric as

$$h_{ij} = e^{2\alpha(x^i,t)}\tilde{h}_{ij} = a^2(t)e^{-2\psi(x^i,t)}\tilde{h}_{ij}\,, \tag{3.7.15}$$

where $a(t)$ is the (homogeneous) global scale factor, and ψ is a perturbation to the scale factor, the curvature perturbation.

We assume that $\det \tilde{h}_{ij} = 1$, and we can factor the $\tilde{h}$ matrix as

$$\tilde{h} = \mathbb{1}e^H \tag{3.7.16}$$

where $\mathbb{1}$ is the unit matrix, and the traceless matrix H_{ij} is a perturbation.

Since the metric of our local observable universe, in the limit $\varepsilon \to 0$, should reduce to the FRW metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j\,, \tag{3.7.17}$$

we can infer some constraints on the metric components N^i, $\tilde{\gamma}_{ij}$.

First, we note that $N^i = O(\varepsilon)$ ($O(\varepsilon)$ means that N^i is *at least* of order ε). Moreover, we have $\dot{\tilde{h}}_{ij} = O(\varepsilon)$, but it can be shown that in Einstein gravity this is a decaying mode, so we require $\dot{\tilde{h}}_{ij} = O(\varepsilon^2)$.

Therefore, keeping terms up to first order in ε, the line element (3.7.13) can be simplified:

$$ds^2 = -N^2 dt^2 + 2N_i dt dx^i + \gamma_{ij}dx^i dx^j\,. \tag{3.7.18}$$

Energy conservation We expect that the energy-momentum tensor will have the perfect fluid form

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + g_\mu \nu P\,. \tag{3.7.19}$$

We must choose a threading: we choose spatial coordinates that comove with the fluid, such that the threads $x^i = \text{const}$ are the integral curves of the 4-velocity u^μ (the usual comoving worldines). Hence

$$v^i = \frac{dx^i}{dt} = \frac{u^i}{u^0} = 0\,. \tag{3.7.20}$$

The components of the 4-velocity in these coordinates will be

$$u^\mu = \left(\frac{1}{\sqrt{N^2 - N^i N_i}}, 0\right) = \left(\frac{1}{N}, 0\right) + O(\varepsilon^2) \tag{3.7.21}$$

$$u_\mu = \left(-\sqrt{N^2 - N^i N_i}, \frac{N_i}{\sqrt{N^2 - N^k N_k}}\right) = \left(-N, \frac{N^i}{N}\right) + O(\varepsilon^2) \,. \tag{3.7.22}$$

The expansion of u^μ is given by

$$\begin{aligned} \vartheta \equiv \nabla_\mu u^\mu =& \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} u^\mu) = \frac{1}{N e^{3\alpha}} \partial_0 (N e^{3\alpha} u^0) = \\ =& \frac{1}{N e^{3\alpha}} \partial_t \left(\frac{N e^{3\alpha}}{\sqrt{N^2 - N^i N_i}}\right) = \frac{3\dot{\alpha}}{N} + O(\varepsilon^2) \,. \end{aligned} \tag{3.7.23}$$

The relation between the time coordinate and the proper time τ along u^μ is

$$\frac{dt}{d\tau} = u^0 = \frac{1}{\sqrt{N^2 - N^i N_i}} \,. \tag{3.7.24}$$

The energy conservation equation along u^μ is

$$\frac{d}{d\tau}\rho + \vartheta(\rho + P) = 0 \tag{3.7.25}$$

and multiplying by u^0 we obtain

$$\dot{\rho} + 3\dot{\alpha}(\rho + P) + O(\varepsilon^2) = 0 \,. \tag{3.7.26}$$

It is important to note that the expansion of n^μ is

$$\vartheta_n \equiv \nabla_\mu n^\mu = \frac{3\dot{\alpha}}{N} - \frac{1}{N e^{3\alpha}} \partial_i (e^{3\alpha} N^i) = \vartheta + O(\varepsilon^2) \,, \tag{3.7.27}$$

so ϑ and ϑ_n are equal at linear order in ε, and this is valid not only for the comoving threading, but for any choice of threading such that $N^i = O(\varepsilon)$.

It is standard use to define a local Hubble parameter by

$$\tilde{H} = \frac{1}{3\vartheta_n} = \frac{1}{N}(H - \dot{\psi}) + O(\varepsilon^2) \,. \tag{3.7.28}$$

Evolution of the curvature perturbation Now we are ready to study the evolution of the curvature perturbation. We start by considering the continuity equation

$$\frac{d}{d\tau}\rho = -3\tilde{H}(\rho + P) + O(\varepsilon^2) = -3\frac{1}{N}(H - \dot{\psi})(\rho + P) + O(\varepsilon^2) \,. \tag{3.7.29}$$

Multiplying by N, we obtain

$$\dot{\psi} = H + \frac{1}{3}\frac{\dot{\rho}}{\rho + P} + O(\varepsilon^2) \tag{3.7.30}$$

Now we choose the uniform energy density slicing, denoting ψ by $-\zeta$ on this slicing.

If P is a unique function of ρ to first order in ε, that is, we have an adiabatic perturbation, then $\dot{\psi}$ is spatially homogeneous to first order.

Since ψ is supposed to vanish at some position in our observable universe (for example at our position), then we have

$$\frac{1}{3}\frac{\dot{\rho}}{\rho + P} = -H \tag{3.7.31}$$

and so ψ is time independent to first order on uniform energy slices:

$$\dot{\psi} = -\dot{\zeta} = O(\varepsilon^2)\,. \tag{3.7.32}$$

Since ζ is the non-linear generalization of the curvature perturbation defined on the uniform energy density slicing, we conclude that at any order in standard perturbation theory ζ is conserved if non-adiabatic pressure perturbations are not present.

Chapter 4

Production and evolution of non linearities

So far, we have developed all the tools we need for the study of production and evolution of non linear perturbations. In this Chapter we start by defining what is meant by non-Gaussian perturbations and how they can be characterized phenomenologically. Then we present a detailed computation of the level of primordial non-Gaussianity expected in three different scenarios for the generation of cosmological perturbations. Finally, we follow the post-inflationary evolution of the non-linear perturbations on large scales and we obtain the level of non-Gaussianity expected in the CMB temperature anisotropies neglecting all the integrated effects deriving from gravitational effects after recombination. For a review, see [18].

4.1 Gaussian and non-Gaussian perturbations

We have seen that a free light field during inflation produces vacuum fluctuations, which are then stretched and amplified on superhorizon scales.

Assuming the field is free, we quantize it by expanding in Fourier modes: since this modes are independent of each other, by the central limit theorem the fluctuations in real space obey Gaussian statistics (actually, for an harmonic oscillator every mode has a Gaussian statistics, but this is an accident with no deep meaning).

If we have a generic perturbation which is related to $\delta\varphi$ by a linear relation, $g(k,t) = T(k,t)\delta\varphi(k,t_0)$ where $T(k,t)$ is called the (linear) transfer function, also g will obey a Gaussian statistics.

So, in linear perturbation theory, if we assume the standard scenario of generation of the perturbations, with the inflaton a free field, both the curvature perturbation and the large-scale CMB anisotropy will obey Gaussian statistics.

This is considered another of the generic predictions of inflation. In fact, it has been verified that the CMB data are consistent with Gaussianity since the first measurement by the *COBE* DMR, but it is much more interesting to consider the small deviation from a Gaussian statistics.

Non Gaussianity is a very important observable, since different models of inflation predict different levels of non Gaussianity, and we could be able to constrain models of inflation, or at least to rule out a part of them.

We expect deviations from a purely Gaussian statistics if we take into account non linear relations between CMB anisotropies and original quantum fluctuations.

To be more precise, we can identify different sources of non Gaussianity [19].

First, there could be self-interactions of the inflaton field, or interactions between different fields in multiple field models (possibly producing also non-adiabatic perturbations), that make original fluctuations non Gaussian of their own.

Second, we have a non linear relation between the primordial curvature perturbation and the field fluctuations, and then a nonlinear evolution of these large-scale perturbations, due to the non linear nature of Einstein equations, which appear simply by going to second order in the perturbations.

Finally, with a second order analysis, we find a non linear relation between the primary temperature anisotropies and the large-scale gravitational potentials at the moment of recombination (generalization of the Sachs-Wolfe effect), and non linear contributions to the integrated Sachs-Wolfe effect.

Clearly, since perturbations are tiny, we expect that second-order effects will be suppressed with respect to first-order ones. However, with the present (and near future) generation of CMB satellite experiment (*WMAP* and *Planck*) we will be able to constrain the level of non Gaussianity in the CMB anisotropies.

To compare theory with observations, it is not sufficient to compute the primordial level of non Gaussianity for a given model of inflation, and then use the linear results to obtain the anisotropies; instead, one must carefully take into account all the second order effects listed above.

4.1.1 Observables

We have already talked about the power spectrum of the perturbation, as a quantity characterizing a given perturbation. If the perturbation obeys a Gaussian statistics, its power spectrum (related to the variance) suffices to characterize the perturbation, since a Gaussian PDF has only two parameters. So, all the odd correlation functions are exactly zero, and the even correlation functions can be written in terms of the two-point function, or in other words their connected part is zero.

If, however, we have a quantity which is not Gaussian distributed, its higher order connected correlation functions do not vanish: so, these are good observables to quantify the level of non Gaussianity.

In particular, the three- and four-point correlation functions of the CMB anisotropies are used, or better their expansion in spherical harmonics, the angular bispectrum and trispectrum.

4.1.2 Parametrization of non-Gaussianity

A phenomenological way of parametrizing the level of non-Gaussianity for a given perturbation is to introduce a non-linearity parameter f_{NL} through the equation [20]

$$\Phi = \Phi_L + f_{NL} \star \left(\Phi^2 - \left\langle \Phi^2 \right\rangle\right) \tag{4.1.1}$$

where Φ_L is the perturbation at linear order, which is a Gaussian distributed quantity, and the $\star$ product reminds the fact that the non-linearity parameter can have a non-trivial scale dependence.

This way we have a definite adimensional parameter that can be measured or at least constrained by CMB measurements, and that can be calculated through a second-order analysis of the cosmological perturbations.

The actual limits on this non linearity parameter coming from the three year *WMAP* data are $-54 < f_{NL} < 114$, at 95%CL [21]; with *Planck* it is expected to obtain $f_{NL} < 5$.

The bispectrum can be simply computed in terms of f_{NL}. Explicitly, we write the gravitational potential in momentum space in the form

$$\phi(\mathbf{k}) = \phi^{(1)}(\mathbf{k}) + \frac{1}{(2\pi)^3} \int d^3k_1\, d^3k_2 \delta^{(3)}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) f^{\phi}_{NL}(\mathbf{k}_1, \mathbf{k}_2)\phi^{(1)}(\mathbf{k}_1)\phi^{(1)}(\mathbf{k}_2) \tag{4.1.2}$$

where f^{ϕ}_{NL} is a momentum-dependent non-linearity parameter, $\phi^{(1)}$ is a Gaussian random field, and a momentum-dependent term must be added to the RHS in order to have $\langle \phi \rangle = 0$.

To leading order in f^{ϕ}_{NL}, the bispectrum reads:

$$\begin{aligned}
&\langle\phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\phi(\mathbf{k}_3)\rangle = \\
=&\frac{1}{(2\pi)^3}\int d^3k_a\, d^3k_b\delta^{(3)}(\mathbf{k}_3-\mathbf{k}_a-\mathbf{k}_b)f^{\phi}_{NL}(\mathbf{k}_a,\mathbf{k}_b)\times \\
&\times\left\langle\phi^{(1)}(\mathbf{k}_1)\phi^{(1)}(\mathbf{k}_2)\phi^{(1)}(\mathbf{k}_a)\phi^{(1)}(\mathbf{k}_b)\right\rangle + \text{cyclic} = \\
=&(2\pi)^3\int d^3k_a\, d^3k_b\delta^{(3)}(\mathbf{k}_3-\mathbf{k}_a-\mathbf{k}_b)f^{\phi}_{NL}(\mathbf{k}_a,\mathbf{k}_b)\times \\
&\times\Big[\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\delta^{(3)}(\mathbf{k}_a+\mathbf{k}_b)P_\phi(k_1)P_\phi(k_a)+ \\
&\quad+\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_a)\delta^{(3)}(\mathbf{k}_2+\mathbf{k}_b)P_\phi(k_1)P_\phi(k_2)+ \\
&\quad+\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_b)\delta^{(3)}(\mathbf{k}_2+\mathbf{k}_a)P_\phi(k_1)P_\phi(k_2)\Big] = \\
=&(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\times\left[2f^{\phi}_{NL}(\mathbf{k}_1,\mathbf{k}_2)P_\phi(k_1)P_\phi(k_2)+\text{cyclic}\right],
\end{aligned} \tag{4.1.3}$$

where we adopt the following definition of the power spectrum:

$$\left\langle\phi^{(1)}(\mathbf{k}_1)\phi^{(1)}(\mathbf{k}_2)\right\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P_\phi(k)\,. \tag{4.1.4}$$

At this point we want to stress that the non-linearity parameter of the gravitational potential does not define the measured level of non-Gaussianity in CMB anisotropies. In fact, as said before, we have to make a further step, and determine how the perturbations in the gravitational potentials translate into second-order fluctuations of the CMB temperature.

4.2 Generation of primordial non-Gaussianity

Now we address the problem of computing the primordial level of non-Gaussianity produced during inflation.

To be specific, we will calculate the bispectrum of scalar and tensor metric perturbations in single field models of inflation, under the slow-roll approximation.

This calculation has been performed for the first time in ref. [22], solving the Einstein and Klein-Gordon equations up to second order; subsequently, the problem was addressed in ref. [23] with a different approach, and taking into account also tensor modes. We will follow the latter, which proceeds by expanding the action up to third order.

4.2.1 Review of first order results

We begin by reviewing briefly in this formalism the quadratic computation, which leads to the usual result for the linear primordial fluctuations.

The starting point is the action of gravity and a scalar field, which has the general form

$$S = \frac{1}{2} \int \sqrt{-g} \left[R - (\nabla\varphi)^2 - V(\varphi) \right] , \tag{4.2.1}$$

where we have set $M_P^{-2} \equiv 8\pi G = 1$, and whose homogeneous solution is the usual FRW metric

$$ds^2 = dt^2 + a^2(t)\delta_{ij}dx^i dx^j \, . \tag{4.2.2}$$

The scalar field is a function of time only, and a and φ obey the equations

$$3H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \tag{4.2.3}$$

$$\dot{H} = -\frac{1}{2}\dot{\varphi}^2 \tag{4.2.4}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0 \tag{4.2.5}$$

where the three are not all independent.

As usual, we define the slow-roll parameters

$$\varepsilon \equiv \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 \simeq \frac{1}{2M_P^2}\frac{\dot{\varphi}^2}{H^2} \tag{4.2.6}$$

$$\eta \equiv M_P^2 \frac{V''}{V} \simeq -\frac{\ddot{\phi}}{H\dot{\phi}} + \frac{1}{2M_P^2}\frac{\dot{\varphi}^2}{H^2} \tag{4.2.7}$$

and we use the slow-roll approximation $\varepsilon, \eta \ll 1$.

To study the small fluctuations around the background metric and scalar field, we find it convenient to use the ADM formalism [24]. We write the metric in the form

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) \tag{4.2.8}$$

and the action reads

$$S = \frac{1}{2}\int \sqrt{h}\left[N\,{}^{(3)}R + \frac{1}{N}\left(E_{ij}E^{ij} - E^2\right) - 2NV + \right.$$
$$\left. + \frac{1}{N}\left(\dot{\varphi} - N^i\partial_i\varphi\right)^2 - Nh^{ij}\partial_i\varphi\partial_j\varphi\right] \tag{4.2.9}$$

where

$$E_{ij} = \frac{1}{2}\left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right) = NK_{ij} \, , \quad E = E^i_{\ i} \, , \tag{4.2.10}$$

${}^{(3)}R$ denotes the intrinsic spatial curvature on the fixed-time hypersurfaces, h_{ij} is the three-metric on these hypersurfaces and ∇_i is the spatial covariant derivative.

In the ADM formalism we can think of h_{ij} and φ as the dynamical variables, while N and N_i are Lagrange multipliers whose time derivatives do not appear in the action.

We will now choose a gauge for h_{ij} and φ: a convenient gauge is

$$\delta\varphi = 0\,, \quad h_{ij} = a^2\left[(1+2\mathcal{R})\delta_{ij} + \gamma_{ij}\right]\,, \quad \partial^i\gamma_{ij} = 0\,, \quad \gamma^i{}_i = 0 \tag{4.2.11}$$

where $\mathcal{R}$ and γ are first order quantities. This gauge is the so-called comoving gauge, and the comoving curvature perturbation is usually denoted by $\mathcal{R}$.

In order to find the action for our degrees of freedom, $\mathcal{R}$ and γ, we just solve for N and N^i through their equations of motion (the hamiltonian and momentum constraints) and plug the result back in the action.

The constraints are

$$^{(3)}R - 2v - \frac{1}{N^2}\left(E_{ij}E^{ij} - E^2\right) - \frac{\dot{\varphi}^2}{N^2} = 0 \tag{4.2.12}$$

$$\nabla_i\left[\frac{1}{N}\left(E^i{}_j - E\delta^i{}_j\right)\right] = 0\,. \tag{4.2.13}$$

If we set $N^i = \partial^i\omega + N^i_T$, with $\partial_i N^i_T = 0$ and $N = 1 + N_1$, the solutions to these equations are

$$N_1 = \frac{\dot{\mathcal{R}}}{H}\,, \quad N^i_T = 0\,, \quad \omega = -\frac{\mathcal{R}}{a^2H} + \chi\,, \quad \partial^2\chi = \frac{\dot{\varphi}^2}{2H^2}\dot{\mathcal{R}}\,. \tag{4.2.14}$$

Now we can replace this expressions in the action and expand it up to quadratic terms. For this purpose it is not necessary to compute N or N^i up to second order, for the second order term in these variables will be multiplying the constraints $\frac{\partial\mathcal{L}}{\partial N}$ and $\frac{\partial\mathcal{L}}{\partial N^i}$ evaluated to zeroth order, which vanish since the zeroth order solutions obey the equations of motion.

The expansion up to second order, considering only $\mathcal{R}$, gives

$$\begin{aligned} S =& \frac{1}{2}\int a e^{\mathcal{R}}\left(1 + \frac{\dot{\mathcal{R}}}{H}\right)\left[-4\partial^2\mathcal{R} - 2(\partial\mathcal{R})^2 - 2Va^2e^{2\mathcal{R}}\right] + \\ &+ a^3e^{3\mathcal{R}}\frac{1}{\left(1+\frac{\dot{\mathcal{R}}}{H}\right)}\left[-6(H+\dot{\mathcal{R}})^2 + \dot{\varphi}^2\right] \end{aligned} \tag{4.2.15}$$

neglecting a total derivative linear in χ.

After integrating by parts some of the terms and using the background equations of motion we find

$$S = \frac{1}{2}\int dt\, d^3x \frac{\dot{\varphi}^2}{H^2}\left[a^3\dot{\mathcal{R}}^2 - a(\partial\mathcal{R})^2\right]\,, \tag{4.2.16}$$

which shows that the action is suppressed by the slow-roll parameter ε. This action describes a free field, so we can quantize $\mathcal{R}$ expanding in Fourier modes:

$$\mathcal{R}(t,x) = \int \frac{d^3k}{(2\pi)^3}\mathcal{R}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{4.2.17}$$

The equation of motion of the action (4.2.16) is

$$a^3\varepsilon\ddot{\mathcal{R}}_{\mathbf{k}} + \dot{\mathcal{R}}_{\mathbf{k}}\frac{da^3\varepsilon}{dt} + \varepsilon a k^2\mathcal{R}_{\mathbf{k}} = 0\,, \tag{4.2.18}$$

and we write $\mathcal{R}_{\mathbf{k}}$ as $\mathcal{R}_{\mathbf{k}}(t) = \mathcal{R}^{cl}_k(t)a^\dagger_{\mathbf{k}} + \mathcal{R}^{cl*}_k a_{-\mathbf{k}}$ where $a^\dagger$, a are the standard creation and annihilation operators, and $\mathcal{R}^{cl}$ are the solutions of the equation of motion.

We have the well-known result

$$\langle\mathcal{R}_{\mathbf{k}}(t)\mathcal{R}_{\mathbf{k}'}(t)\rangle \simeq (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\frac{1}{2k^3}\left(\frac{H^2}{M_P\dot{\varphi}}\right)^2_* \tag{4.2.19}$$

where the $_*$ means that we must evaluate these quantities at horizon crossing.

If we focus on the terms quadratic in γ in the action, we find

$$S = \frac{1}{8}\int\left[a^3\dot{\gamma}_{ij}\dot{\gamma}^{ij} - a\partial^l\gamma^{ij}\partial_l\gamma_{ij}\right]\,. \tag{4.2.20}$$

As already noted, the γ describe gravitational waves: we can expand them in plane waves with definite polarization tensors

$$\gamma_{ij} = \int\frac{d^3k}{(2\pi)^3}\sum_{s=\pm}\varepsilon^s_{ij}(k)\gamma^s_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4.2.21}$$

with $\varepsilon^i_{;i} = k^i\varepsilon_{ij} = 0$ and $\varepsilon^s_{ij}\varepsilon^{s'}_{ij} = 2\delta_{ss'}$. Therefore, for each polarization we have essentially the equation of motion for a massless scalar field: as discussed before, the solutions become constant after horizon crossing. Computing the two-point function just after horizon crossing we find

$$\left\langle\gamma^s_{\mathbf{k}}\gamma^{s'}_{\mathbf{k}'}\right\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\delta_{ss'}\frac{1}{k^3}\left(\frac{H}{M_P}\right)\,, \tag{4.2.22}$$

which is the power spectrum of the stochastic background of gravitational waves produced during inflation.

4.2.2 Second order results

Now we compute the cubic terms in the action, in order to give an estimate of the three point functions.

Up to second order, we fix the gauge as

$$\delta\varphi = 0\,,\quad h_{ij} = a^2e^{2\mathcal{R}}\left(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma^l_i\gamma_{lj}\right) \tag{4.2.23}$$

where $\gamma^i_{;i} = \partial^i \gamma_{ij} = 0$.

It is possible to show that $\mathcal{R}$ and γ are constant outside the horizon, to all orders. For this purpose, we can expand the action to first order in derivatives but to all orders in powers of the fields. From the structure of the hamiltonian and momentum constraints, we see that we can assume $N = 1 + \delta N$, where δN has an expansion in derivatives starting with a first order term; and we can assume that N^i is of zeroth order in derivatives, while $\nabla_i N_j$ are of first order in derivatives.

Expanding the hamiltonian constraint to first order in derivatives we find

$$\delta N = \frac{H}{V}\left(3\dot{\mathcal{R}} - \nabla_i N^i\right) . \tag{4.2.24}$$

On a solution of the hamiltonian constraint, the action reads

$$\begin{aligned} S &= \int \sqrt{h} N \left({}^{(3)}R - 2V\right) = \int \sqrt{h}\,(-2V - 2V\delta N) = \\ &= \int a^3 e^{3\mathcal{R}} \left(-6H^2 + \dot{\varphi}^2 - 6H\dot{\mathcal{R}}\right) = -2 \int \frac{d}{dt}\left(a^3 H e^{3\mathcal{R}}\right) \end{aligned} \tag{4.2.25}$$

where we have neglected the term involving ${}^{(3)}R$ because it is of second order in derivatives, we integrated by parts the term involving $\nabla_i N^i$ and we used the background equations. Therefore, on a solution of the background equations, the Lagrangian is a total derivative and can be ignored.

In conclusion, both $\mathcal{R}$ and γ are constant outside the horizon. The reason, as already noted with the gradient expansion, is that outside the horizon we can neglect spatial derivatives. Since we also showed that the expansion in time derivatives starts at second order, constant $\mathcal{R}$ and γ are solutions of the equations of motion (to all orders in powers of the fields) outside the horizon.

We now expand the action up to cubic order in $\mathcal{R}$. It suffices to know the constraints N and N^i only up to first order, because the second order terms would be multiplying the constraints evaluated at first order which vanish due to the first order expressions for N and N^i. Up to total derivatives, we find

$$\begin{aligned} S = \int a e^{\mathcal{R}} \left(1 + \frac{\dot{\mathcal{R}}}{H}\right)\left[-2\partial^2\mathcal{R} - (\partial\mathcal{R})^2\right] + a^3 e^{3\mathcal{R}} \frac{\dot{\varphi}^2}{2H^2}\dot{\mathcal{R}}^2 \left(1 - \frac{\dot{\mathcal{R}}}{H}\right) + \\ + a^3 e^{3\mathcal{R}} \left[\frac{1}{2}\left((\partial_i\partial_j\psi)(\partial^i\partial^j\psi) - (\partial^2\psi)^2\right)\left(1 - \frac{\dot{\mathcal{R}}}{H}\right) - 2(\partial_i\psi)(\partial^i\mathcal{R})\partial^2\psi\right] \end{aligned} \tag{4.2.26}$$

where we expand the exponentials to keep only terms of up to third order in $\mathcal{R}$.

It is not obvious from this form of the action that the effective cubic interaction is of second order in slow-roll parameters. To show this, we manipulate the action, doing some integration by parts and dropping total derivatives to obtain the form

$$
\begin{aligned}
S_3 = \int & \frac{1}{4}\left(\frac{\dot\varphi}{H}\right)^4 \left[a^3\dot{\mathcal{R}}^2\mathcal{R} + a\mathcal{R}(\partial\mathcal{R})^2\right] - \left(\frac{\dot\varphi}{H}\right)^2 a^3\dot{\mathcal{R}}\,\partial_i\chi\partial^i\mathcal{R} + \\
& - \frac{1}{16}\left(\frac{\dot\varphi}{H}\right)^6 a^3\dot{\mathcal{R}}^2\mathcal{R} + \left(\frac{\dot\varphi}{H}\right)^2 a^3\dot{\mathcal{R}}\mathcal{R}^2 \frac{d}{dt}\left[\frac{1}{2}\frac{\ddot\varphi}{H\dot\varphi} + \frac{1}{4}\frac{\dot\varphi^2}{H^2}\right] + \\
& + \frac{1}{4}\left(\frac{\dot\varphi}{H}\right)^2 a^3\mathcal{R}(\partial^i\partial^j\chi)(\partial_i\partial_j\chi) + f(\mathcal{R})\left.\frac{\delta\mathcal{L}}{\delta\mathcal{R}}\right|_{(1)}
\end{aligned}
\qquad (4.2.27)
$$

where the leading terms are of order ε^2, as expected; χ is given by eq. (4.2.14) and is of order ε, and the last term is a term proportional to the first order equations of motion (4.2.18). It can be removed by performing the following field redefinition:

$$
\begin{aligned}
\mathcal{R} = \mathcal{R}_n - f(\mathcal{R}_n) = \mathcal{R}_n + \left(\frac{1}{2}\frac{\ddot\varphi}{H\dot\varphi} + \frac{1}{4}\frac{\dot\varphi^2}{H^2}\right)\mathcal{R}^2 + \frac{1}{H}\dot{\mathcal{R}}\mathcal{R} - \frac{1}{4}\frac{(\partial\mathcal{R})^2}{(aH)^2} + \\
+ \frac{1}{4}\frac{1}{(aH)^2}\partial^{-2}\left[\partial^i\partial^j(\partial_i\mathcal{R}\,\partial_j\mathcal{R})\right] + \frac{1}{2H}\partial^i\chi\,\partial_i\mathcal{R} - \frac{1}{2H}\partial^{-2}\left[\partial^i\partial^j(\partial_i\chi\,\partial_j\mathcal{R})\right].
\end{aligned}
\qquad (4.2.28)
$$

After this field redefinition, the action is written in terms of $\mathcal{R}_n$ and is given by eq. (4.2.27) without the last term.

However, the field redefinition is important for our calculation, because $\mathcal{R}$ remains constant outside the horizon, while $\mathcal{R}_n$ does not (this is apparent from the field redefinition).

In order to perform the computation of the three point function we will use a variable $\mathcal{R}_c$ defined through

$$
\mathcal{R} = \mathcal{R}_c + \frac{1}{2}\frac{\ddot\varphi}{H\dot\varphi}\mathcal{R}_c^2 + \frac{1}{8}\frac{\dot\varphi^2}{H^2}\mathcal{R}_c^2 + \frac{1}{4}\frac{\dot\varphi^2}{H^2}\partial^{-2}(\mathcal{R}_c\partial^2\mathcal{R}_c) + \dots \qquad (4.2.29)
$$

where the dots indicate terms that vanish outside the horizon or are higher order in the slow-roll parameters. In terms of $\mathcal{R}_c$, the action reads

$$
S_3 = \int \left(\frac{\dot\varphi^2}{H^2}\right)^2 a^5 H\dot{\mathcal{R}}_c^2\partial^{-2}\dot{\mathcal{R}}_c + \dots \qquad (4.2.30)
$$

where the dots again indicate terms of higher order in slow-roll.

For two scalars and a graviton, we find

$$
\begin{aligned}
S = \int & -2\frac{a}{H}\gamma_{ij}\partial^i\dot{\mathcal{R}}\,\partial^j\mathcal{R} - a\gamma_{ij}\partial^i\mathcal{R}\,\partial^j\mathcal{R} - \frac{1}{2}a^3\left(3\mathcal{R} - \frac{\dot{\mathcal{R}}}{H}\right)\dot\gamma_{ij}\partial^i\partial^j\psi + \\
& + \frac{1}{2}a^3(\partial_l\gamma_{ij})(\partial^i\partial^j\psi)(\partial^l\psi).
\end{aligned}
\qquad (4.2.31)
$$

To understand the dependence on the slow-roll parameters, we integrate by parts to obtain

$$S = \int \frac{1}{2}\frac{\dot{\varphi}^2}{H^2} a\gamma_{ij}\,\partial^i\mathcal{R}\partial^j\mathcal{R} + \frac{1}{4}a^3\left(\partial^2\gamma_{ij}\right)\partial^i\chi\partial^j\chi + \frac{1}{4}\frac{\dot{\varphi}^2}{H^2}a^3\dot{\gamma}_{ij}\partial^i\mathcal{R}\partial^j\mathcal{R} + \\ + \hat{f}(\mathcal{R},\gamma)\left.\frac{\delta\mathcal{L}}{\delta\mathcal{R}}\right|_{(1)} + \hat{f}_{ij}(\mathcal{R})\left.\frac{\delta\mathcal{L}}{\delta\gamma_{ij}}\right|_{(1)} \tag{4.2.32}$$

where the first terms are of second order in slow-roll parameters, and the last terms can be removed by an appropriate field redefinition, as before. It turns out that these field redefinitions are not important after horizon crossing and hence are not important for our computation.

For two gravitons and a scalar, we find

$$S = \int \frac{1}{16}\frac{\dot{\varphi}^2}{H^2}\left[a^3\mathcal{R}\dot{\gamma}_{ij}\dot{\gamma}^{ij} + a\mathcal{R}\partial_l\gamma_{ij}\partial^l\gamma^{ij}\right] - \frac{1}{4}a^3\dot{\gamma}_{ij}\partial_l\gamma^{ij}\partial_l\chi - \mathcal{R}\dot{\gamma}_{ij}\left.\frac{\delta\mathcal{L}}{\delta\gamma_{ij}}\right|_{(1)} + \dots \tag{4.2.33}$$

and as usual the last term can be removed by a field redefinition.

The action is rather similar to that for three scalars, and so to perform the computation we do the further field redefinition

$$\mathcal{R} = \mathcal{R}_c - \frac{1}{32}\gamma_{ij}\gamma^{ij} + \frac{1}{16}\partial^{-2}\left(\gamma_{ij}\partial^2\gamma^{ij}\right) + \dots\,. \tag{4.2.34}$$

The action becomes

$$S = \int \frac{1}{4}\frac{\dot{\varphi}^2}{H^2}a^5 H\dot{\gamma}_{ij}\dot{\gamma}^{ij}\partial^{-2}\dot{\mathcal{R}}_c + \dots \tag{4.2.35}$$

Finally, the action for three gravitons receives the only contribution from the term

$$S = \frac{1}{2}\int a^2\left(\hat{R} + E^i_{\ j}E^j_{\ i}\right) \tag{4.2.36}$$

where $\hat{R}$ is the spatial curvature corresponding to the metric $\hat{h}_{ij}$.

4.2.3 Computation of three point functions

Now we are able to compute the three point functions, using the interaction Lagrangians given above. It is important to note that these are expectation values of product of fields at a fixed time, and not a scattering amplitude: we have to compute [25]

$$\left\langle\mathcal{R}^3(t)\right\rangle = \left\langle\left[\overline{T}\exp\left(i\int_{t_0}^t H_I(t')dt'\right)\right]\mathcal{R}_I^3(t)\left[T\exp\left(-i\int_{t_0}^t H_I(t')dt'\right)\right]\right\rangle \tag{4.2.37}$$

where the I stands for interaction picture, T and $\overline{T}$ denote respectively time-ordering and anti-time-ordering, and t_0 is some early time.

To first order this gives

$$\left\langle \mathcal{R}^3(t) \right\rangle = -i \int_{t_0}^{t} dt' \left\langle \left[\mathcal{R}_I^3(t), H_I(t') \right] \right\rangle \tag{4.2.38}$$

where for the cubic terms $H_I = -\mathcal{L}_I$ after removing terms proportional to the equations of motion.

We have also to note that we want to compute the expectation value in the interacting vacuum, and not in the free vacuum.

In Minkowski space this is taken into account by deforming the t' integration contour so that it includes some evolution in Euclidean time.

Since in our case at early times every mode is well inside the horizon, we can approximate the space as flat and so the true vacuum will be selected as in Minkowski space: this is the Hartle-Hawking prescription for the vacuum, and in practice corresponds to a choice of the integration contour for the integral.

So, we are left to evaluate the integral (4.2.38) with the interaction Lagrangians calculated above, on the solutions of the background equations of motion. However, we have not the explicit solution for a generic potential, so we need to approximate the integral in some way.

It is convenient to consider separately the region outside the horizon, the region around horizon crossing and the region deep inside the horizon.

Deep inside the horizon, the fields oscillate rapidly, and the contribution to the integral vanishes after the analytic continuation to Euclidean time.

Outside the horizon, we know that $\mathcal{R}$ and γ remain constant, so this contribution to the integral vanishes too.

In the region near horizon crossing, we can approximate the solutions by those for a massless field in de Sitter space, and the dependence on the potential is taken into account by using the leading order terms in slow-roll of the action.

For three scalars, we note that with a field redefinition of the form $\mathcal{R} = \mathcal{R}_c + \lambda \mathcal{R}_c^2$ the three point function can be written as

$$\begin{aligned}\langle \mathcal{R}(x_1)\mathcal{R}(x_2\mathcal{R}(x_3)\rangle = &\langle \mathcal{R}_c(x_1)\mathcal{R}_c(x_2\mathcal{R}_c(x_3)\rangle + \\ &+ 2\lambda \left[\langle \mathcal{R}_c(x_1)\mathcal{R}_c(x_2)\rangle \langle \mathcal{R}_c(x_1)\mathcal{R}_c(x_3)\rangle + \text{cyclic} \right] .\end{aligned} \tag{4.2.39}$$

For the first term we use the action (4.2.30), evaluating it in de Sitter space with parameters corresponding to those of horizon exit:

$$\begin{aligned}\langle \mathcal{R}_c\mathcal{R}_c\mathcal{R}_c\rangle =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,\frac{1}{\prod_i(2k_i^3)}\,\frac{H_*^6}{\dot\varphi_*^2}\times\\ &\times\int_{-\infty}^{0} d\tau\, k_1^2\, k_2^2 e^{ik\tau} + \text{perm} + \text{c.c.} =\\ =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{1}{\prod_i(2k_i^3)}\,\frac{H_*^6}{\dot\varphi_*^2}\,4\frac{k_1^2k_2^2+k_1^2k_3^2+k_2^2k_3^2}{k}\end{aligned} \tag{4.2.40}$$

where $k = k_1 + k_2 + k_3$, and we have chosen the contour of integration so that $\tau \to \tau + i\varepsilon|\tau|$ for large $|\tau|$.

After adding the contribution of the second term, the final result for the three point function is

$$\langle \mathcal{R}_{\mathbf{k}_1}\mathcal{R}_{\mathbf{k}_2}\mathcal{R}_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{1}{\prod_i(2k_i^3)}\left(\frac{H_*^4}{M_P^2\dot\varphi_*^2}\right)^2\mathcal{A}_* \tag{4.2.41}$$

where the star indicates evaluation at horizon crossing and

$$\mathcal{A}_* = 2\frac{\ddot\varphi_*}{H_*\dot\varphi_*}\sum_i k_i^3 + \frac{\dot\varphi_*^2}{H_*^2}\left[\frac{1}{2}\sum_i k_i^3 + \frac{1}{2}\sum_{i\neq j} k_i k_j^2 + \frac{4}{k}\sum_{i>j} k_i^2 k_j^2\right]. \tag{4.2.42}$$

From this result, it is apparent that in the standard scenario of single field inflation the level of primordial non-Gaussianity is tiny, being proportional to ε.

We can define a momentum-dependent non-linearity parameter as

$$f_{NL} \simeq -\frac{5}{3}\frac{\mathcal{A}}{4\sum_i k_i^3} = -\frac{5}{12}\left[2\frac{\ddot\varphi_*}{H_*\dot\varphi_*} + \frac{\dot\varphi_*^2}{H_*^2}(2+f(k))\right] = \frac{5}{12}\left(n_s + f(k)n_t\right) \tag{4.2.43}$$

where n_s and n_t are the scalar and tensor tilt, respectively, and $f(k)$ is a function of the shape of the triangle formed by the $\mathbf{k}_i$ and has a range of values $0 \le f(k) \le \frac{5}{6}$.

For two scalars and a graviton, the calculation is similar. Since in this case field redefinitions are not important at late times, as interaction Lagrangian we can take the first term of eq. (4.2.32). This gives

$$\left\langle \gamma^s_{\mathbf{k}_1}\mathcal{R}_{\mathbf{k}_2}\mathcal{R}_{\mathbf{k}_3}\right\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{1}{\prod_i(2k_i^3)}\frac{H_*^4}{M_P^4}\frac{H_*^2}{\dot\varphi_*^2}\varepsilon^s_{ij}k_2^i k_3^j 4I \tag{4.2.44}$$

where I is

$$\begin{aligned}I =&\Re\left[-i\int_{-\infty}^{0}\frac{d\tau}{\tau^2}(1-ik_1\tau)(1-ik_2\tau)(1-ik_3\tau)e^{ik\tau}\right] =\\ =&-k + \frac{\sum_{i>j}k_ik_j}{k} + \frac{k_1k_2k_3}{k^2}\end{aligned} \tag{4.2.45}$$

where the divergence in the integral for $\tau \to 0$ is purely imaginary so I is finite with our choice of contour.

We note that the dependence on the slow-roll parameters is the same as in the three scalar case, so that the two correlation functions are of the same order of magnitude. Since after horizon reentry the amplitude of gravitational waves begins to decay we expect for high l the three $\mathcal{R}$ correlators to dominate.

For two gravitons and a scalar, we use the field redefinition (4.2.34) and the interaction (4.2.35) to find

$$\begin{aligned}\left\langle \mathcal{R}_{\mathbf{k}_1}\gamma^{s_2}_{\mathbf{k}_2}\gamma^{s_3}_{\mathbf{k}_3}\right\rangle =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{1}{\prod_i(2k_i^3)}\frac{H_*^4}{M_P^4}\times\\ &\times\left[-\frac{1}{4}k_1^3+\frac{1}{2}k_1(k_2^2+k_3^2)+4\frac{k_2^2k_3^2}{k}\right]\varepsilon^{s_2}_{ij}\varepsilon^{s_3}_{ij}\,.\end{aligned} \tag{4.2.46}$$

Finally, in the case of three gravitons, we find

$$\left\langle \gamma^{s_1}_{\mathbf{k}_1}\gamma^{s_2}_{\mathbf{k}_2}\gamma^{s_3}_{\mathbf{k}_3}\right\rangle = -(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{1}{\prod_i(2k_i^3)}\frac{H_*^4}{M_P^4}4I\left(\varepsilon^{s_1}_{ii'}\varepsilon^{s_2}_{jj'}\varepsilon^{s_3}_{ll'}t_{ijl}t_{i'j'l'}\right) \tag{4.2.47}$$

where t_{ijl} is given by

$$t_{ijl} = k_1^l\delta_{ij} + k_2^i\delta_{jl} + k_3^j\delta_{il}\,. \tag{4.2.48}$$

4.3 Alternative scenarios

So far, we have analyzed the simplest model of inflation, in which one considers a single scalar field slowly rolling down its potential. The mechanism of production of the primordial perturbations in this model is often referred to as the standard scenario, in which, as we have just seen, the level of primordial non-Gaussianity is negligible.

However, the standard scenario is a non-realistic model: therefore, different models of inflation have been studied and different scenarios to produce the primordial perturbations have been proposed.

Here we report two of them, computing the curvature perturbation predicted up to second order [26], thus estimating the level of non-Gaussianity.

4.3.1 The curvaton scenario

In the curvaton scenario (see [27, 28]), the cosmological perturbations are generated from the fluctuations of a light scalar field σ different from the inflaton, in the case where the inflaton perturbations are negligible. The scalar field is subdominant during inflation and very weakly coupled, so its fluctuations are initially of isocurvature type. A curvature perturbation is sourced on large scales by the non-adiabatic pressure perturbation (for the conversion of isocurvature perturbations into curvature ones see ref. [29]), and it becomes relevant when the energy density of the curvaton field is a significant fraction of the total energy density. This happens after the end of inflation, when the curvaton field starts to oscillate around the minimum of its potential once its mass has dropped below the Hubble rate. Finally the curvaton field is supposed to decay completely into thermalised radiation thus generating a final adiabatic perturbation, and from now on the standard evolution takes place.

First order

We suppose that the curvaton is an almost free scalar field with a small effective mass $m_\sigma^2 = \left|\frac{\partial^2 V}{\partial\sigma^2}\right| \ll H_I^2$.

We expand the curvaton field as $\sigma(\tau, \mathbf{x}) = \sigma(\tau) + \delta^{(1)}\sigma(\tau, \mathbf{x})$. The unperturbed field satisfies the equation

$$\sigma'' + 2\mathcal{H}\sigma' + a^2\frac{\partial V}{\partial\sigma} = 0\,, \tag{4.3.1}$$

and the linear fluctuation satisfies on large scales (after horizon exit) the equation

$$\delta^{(1)}\sigma'' + 2\mathcal{H}\delta^{(1)}\sigma' + a^2 m_\sigma^2\delta^{(1)}\sigma = 0\,. \tag{4.3.2}$$

But this is just the equation for a generic scalar field in a quasi-de Sitter stage, so the fluctuations $\delta\sigma$ on large scales will be Gaussian distributed, with a power spectrum

$$\mathcal{P}_{\delta\sigma}(k) \simeq \left(\frac{H_*}{2\pi}\right)^2 \tag{4.3.3}$$

where the $_*$ denotes the epoch of horizon exit, $k = aH$.

After the end of inflation, the inflaton decays into radiation, and the curvaton continues to slowly roll until the Hubble rate becomes comparable to its mass, $H^2 \simeq m_\sigma^2$: at this stage it begins to oscillate around the minimum of its potential, which can be approximated by the quadratic term $V \simeq \frac{1}{2}m_\sigma^2\sigma^2$. In this case we see from eqs. (4.3.1, 4.3.2) that the fractional field perturbation has the constant value

$$\frac{\delta^{(1)}\sigma}{\sigma} = \left(\frac{\delta^{(1)}\sigma}{\sigma}\right)_*\,. \tag{4.3.4}$$

The energy density of the oscillating curvaton field is given by

$$\rho_\sigma(\tau, \mathbf{x}) = m_\sigma^2\tilde{\sigma}^2(\tau, \mathbf{x}) \tag{4.3.5}$$

where $\tilde{\sigma}$ is the amplitude of the oscillation.

We now expand this energy density up to first order to find

$$\rho_\sigma(\tau, \mathbf{x}) = \rho_\sigma(\tau) + \delta\rho_\sigma(\tau\mathbf{x}) = m_\sigma^2\sigma^2 + 2m_\sigma^2\sigma\delta^{(1)}\sigma \tag{4.3.6}$$

and so, using eq. (4.3.4) the relative energy density perturbation on large scales is given by

$$\frac{\delta^{(1)}\rho_\sigma}{\rho_\sigma} = 2\left(\frac{\delta^{(1)}\sigma}{\sigma}\right)_* . \tag{4.3.7}$$

At this stage we have a non-adiabatic pressure perturbation, due to the mixture of matter (the oscillating curvaton field) and radiation. So, the curvature perturbation $\zeta^{(1)}$ evolves in time, until the pressure perturbation becomes adiabatic, at the epoch of curvaton matter domination or after curvaton decay (supposing that radiation thermalizes with the decay products), whichever is earlier.

It is convenient to consider the curvature perturbations $\zeta_i^{(1)}$ associated with radiation and the curvaton field. The total curvature perturbation is given by

$$\zeta^{(1)} = f\zeta_\sigma^{(1)} + (1-f)\zeta_\gamma^{(1)} \tag{4.3.8}$$

where

$$f = \frac{3\rho_\sigma}{4\rho_\gamma + 3\rho_\sigma} \tag{4.3.9}$$

is the relative contribution of the curvaton to the total curvature perturbation.

We now make the approximation of sudden decay of the curvaton field: under this approximation both the curvaton and radiation components satisfy separately the energy conservation equations

$$\rho'_\gamma = -4\mathcal{H}\rho_\gamma\,, \tag{4.3.10}$$

$$\rho'_\sigma = -3\mathcal{H}\rho_\sigma\,, \tag{4.3.11}$$

and the curvature perturbations ζ_i remain constant on superhorizon scales until the curvaton decay.

The evolution of the total curvature perturbation on large scales is given by

$$\zeta^{(1)\prime} = f'\left(\zeta_\sigma^{(1)} - \zeta_\gamma^{(1)}\right) = \mathcal{H}f(1-f)\left(\zeta_\sigma^{(1)} - \zeta_\gamma^{(1)}\right) \tag{4.3.12}$$

from which we read the expression for the non-adiabatic presure perturbation

$$\delta^{(1)}P_{\mathrm{nad}} = \rho_\sigma f(1-f)\left(\zeta_\sigma^{(1)} - \zeta_\gamma^{(1)}\right) . \tag{4.3.13}$$

Now, in this scenario the perturbation in the radiation component is considered to be negligible, ${\zeta^{(1)}}_\gamma \simeq 0$, while the value of ${\zeta^{(1)}}_\sigma$ is fixed by the fluctuations of the curvaton field during inflation, ${\zeta^{(1)}}_\sigma = {\zeta^{(1)}}_{\sigma I}$. So, the total curvature perturbation during the curvaton oscillations is given by

$$\zeta^{(1)} = f{\zeta^{(1)}}_\sigma \,. \tag{4.3.14}$$

From this equation, it is clear that initially, when the curvaton energy density is subdominant, the perturbation ${\zeta^{(1)}}_\sigma$ gives a negligible contribution to the total curvature perturbation, corresponding in fact to an isocurvature perturbation. On the other hand, during the oscillations the curvaton energy density $\rho_\sigma \propto a^{-3}$ increases with respect to the radiation energy density $\rho_\gamma \propto a^{-4}$, and the perturbations in the curvaton field are converted into the total curvature perturbation. After the curvaton decay, the total curvature perturbation remains constant on superhorizon scales at a value fixed by eq. (4.3.14) in the sudden decay approximation:

$$\zeta^{(1)} = f_D{\zeta^{(1)}}_\sigma \,, \tag{4.3.15}$$

where $_D$ stands for the epoch of curvaton decay.

Going beyond the sudden decay approximation, it is possible to introduce a transfer parameter r, defined as

$$\zeta^{(1)} = r{\zeta^{(1)}}_\sigma \tag{4.3.16}$$

where $\zeta^{(1)}$ is evaluated well after the epoch of curvaton decay (when it is a constant) and ${\zeta^{(1)}}_\sigma$ is evaluated well before this epoch. With a numerical study of the perturbation equations it is possible to show that the sudden decay approximation is exact when the curvaton dominates the energy density before it decays, so that $r = 1$, while in the opposite case r is given by

$$r \simeq \left(\frac{\rho_\sigma}{\rho}\right)_D \,. \tag{4.3.17}$$

Second order

We now extend the calculation of the curvature perturbation at second order. It is useful to consider the individual curvature perturbations $\zeta^{(2)}_\sigma$ and $\zeta^{(2)}_\gamma$, to write $\zeta^{(2)}$ as the weighted sum

$$\zeta^{(2)} = f\zeta^{(2)}_\sigma + (1-f)\zeta^{(2)}_\gamma + f(1-f)(1+f)\left(\zeta^{(1)}_\sigma - \zeta^{(1)}_\gamma\right)^2 , \tag{4.3.18}$$

where ${\zeta^{(2)}}_i$ are separately conserved in the sudden decay limit and f is given by eq. (4.3.9).

Therefore $\zeta^{(2)}$ evolves in time according to the equation

$$\zeta^{(2)\prime} = f'\left(\zeta^{(2)}_\sigma - \zeta^{(2)}_\gamma\right) + f'(1-3f^2)\left(\zeta^{(1)}_\sigma - \zeta^{(1)}_\gamma\right)^2 , \tag{4.3.19}$$

from which we can read the expression for the non-adiabatic pressure perturbation at second order

$$\delta^{(2)}P_{\rm nad} = \rho_\sigma(1-f)\left[\left(\zeta_\sigma^{(2)} - \zeta_\gamma^{(2)}\right) + (f^2+6f-1)\left(\zeta_\sigma^{(1)} - \zeta_\gamma^{(1)}\right)^2 + \right.$$
$$\left. +4\zeta_\gamma^{(1)}\left(\zeta_\sigma^{(1)} - \zeta_\gamma^{(1)}\right)\right] . \quad (4.3.20)$$

After the curvaton decay, the second order curvature perturbation remains constant and, in the sudden decay approximation, its value is given by

$$\zeta^{(2)} = f_D\zeta_\sigma^{(2)} + f_D(1-f_D^2)\left(\zeta_\sigma^{(1)}\right)^2 , \quad (4.3.21)$$

where we have used the hypothesis that the perturbation in the radiation energy density produced at the end of inflation is negligible.

Now, since $\zeta^{(r)}$ is a gauge invariant quantity, we choose to work in the spatially flat gauge $\psi^{(r)} = \chi_{ij}^{(r)} = 0$. In this gauge, the gauge-invariant curvature perturbation is indeed expressed as a density perturbation, and we find

$$\zeta_\sigma^{(1)} = \frac{1}{3}\frac{\delta^{(1)}\rho_\sigma}{\rho_\sigma} = \frac{2}{3}\frac{\delta^{(1)}\sigma}{\sigma} = \frac{2}{3}\left(\frac{\delta^{(1)}\sigma}{\sigma}\right)_* \quad (4.3.22)$$

$$\zeta_\sigma^{(2)} = \frac{1}{3}\frac{\delta^{(2)}\sigma}{\sigma} - \left(\zeta_\sigma^{(1)}\right)^2 . \quad (4.3.23)$$

Expanding the curvaton energy density up to second order, we have

$$\begin{aligned}\rho_\sigma(\tau,\mathbf{x}) =& \rho_\sigma(\tau) + \delta^{(1)}\rho_\sigma(\tau\mathbf{x}) + \frac{1}{2}\delta^{(2)}\rho_\sigma(\tau,\mathbf{x}) = \\ =& m_\sigma^2\sigma^2 + 2m_\sigma^2\sigma\delta^{(1)}\sigma + m_\sigma^2\left(\delta^{(1)}\sigma\right)^2 .\end{aligned} \quad (4.3.24)$$

It follows that

$$\frac{\delta^{(2)}\rho_\sigma}{\rho_\sigma} = \frac{1}{2}\left(\frac{\delta^{(1)}\rho_\sigma}{\rho_\sigma}\right)^2 = \frac{9}{2}\left(\zeta_\sigma^{(1)}\right)^2 , \quad (4.3.25)$$

where we have used eq. (4.3.22); hence, from eq. (4.3.23) we find

$$\zeta_\sigma^{(2)} = \frac{1}{2}\left(\zeta_\sigma^{(1)}\right)^2 = \frac{1}{2}\left(\zeta_\sigma^{(1)}\right)_I^2 \quad (4.3.26)$$

and so also $\zeta^{(2)}$ is a conserved quantity, whose value depends on the fluctuations of the curvaton during inflation.

Finally, the second order curvature perturbation produced in the curvaton scenario turns out to be

$$\zeta^{(2)} = f_D\left(\frac{3}{2} - f_D^2\right)\left(\zeta_\sigma^{(1)}\right)^2 , \quad (4.3.27)$$

which shows that the primordial level of non-Gaussianity can be much larger than that of the standard scenario.

4.3.2 The inhomogeneous reheating scenario

In the inhomogeneous reheating (or modulated reheating) scenario (see [30]), the cosmological perturbations are generated during the reheating stage after inflation by means of spatial fluctuations in the inflaton decay rate. In fact, in most realistic models of inflation the coupling of the inflaton to ordinary matter is determined by the vacuum expectation values of other fields in the theory, as in the case of supersymmetric theories or theories inspired by superstrings (for a review, see [31]). If these fields are light, they will fluctuate and their fluctuations lead to spatial fluctuations in the inflaton decay rate, which in turns lead to fluctuations in the reheating temperature, that is, in the radiation energy density.

First order

To study the generation of perturbations in this scenario, we have to consider a system composed of two fluids, the oscillating scalar field φ and the radiation fluid, which have the energy-momentum tensors $T^{\mu\nu}_{(\varphi)}$ and $T^{\mu\nu}_{(\gamma)}$.

The total energy-momentum tensor $T^{\mu\nu} = T^{\mu\nu}_{(\varphi)} + T^{\mu\nu}_{(\gamma)}$ is covariantly conserved, but because of the interaction between the two fluids, the individual energy-momentum tensors are not separately covariantly conserved.

However, introducing the energy-momentum transfer $\hat{Q}^{\nu}_{(\varphi)}$ and $\hat{Q}^{\nu}_{(\gamma)}$ subject to the constraint $\hat{Q}^{\nu}_{(\varphi)} + \hat{Q}^{\nu}_{(\gamma)} = 0$, we can write the equations

$$\nabla_\mu T^{\mu\nu}_{(\varphi)} = \hat{Q}^{\nu}_{(\varphi)} \tag{4.3.28}$$

$$\nabla_\mu T^{\mu\nu}_{(\gamma)} = \hat{Q}^{\nu}_{(\gamma)} \, . \tag{4.3.29}$$

We can decompose the energy momentum transfer as

$$\hat{Q}^{\nu}_{(\varphi)} = -Q_\varphi u^\nu + f^{\nu}_{(\varphi)} \tag{4.3.30}$$

$$\hat{Q}^{\nu}_{(\gamma)} = -Q_\gamma u^\nu + f^{\nu}_{(\gamma)} \tag{4.3.31}$$

where the f^ν are orthogonal to the total velocity of the fluid u^ν.

With this definition, the energy continuity equations for the scalar field and the radiation fluid are

$$u_\nu \nabla_\mu T^{\mu\nu}_{(\varphi)} = Q_\varphi \tag{4.3.32}$$

$$u_\nu \nabla_\mu T^{\mu\nu}_{(\gamma)} = Q_\gamma \, . \tag{4.3.33}$$

where in the case of an oscillating scalar field decaying into radiation the energy transfer coefficients are given by

$$Q_\varphi = -\Gamma \rho_\varphi = -Q_\gamma \, . \tag{4.3.34}$$

The background energy continuity equations read

$$\rho'_\varphi = -3\mathcal{H}(\rho_\varphi + P_\varphi) + aQ^{(0)}_\varphi \tag{4.3.35}$$
$$\rho'_\gamma = -3\mathcal{H}(\rho_\gamma + P_\gamma) + aQ^{(0)}_\gamma \tag{4.3.36}$$

where $Q^{(0)}_\varphi$, $Q^{(0)}_\gamma$ are the background values of the transfer coefficients.

Up to first order, from eqs. (4.3.32-4.3.33) we find

$$\delta^{(1)}\rho_\varphi + 3\mathcal{H}\left(\delta^{(1)}\rho_\varphi + \delta^{(1)}P_\varphi\right) - 3\psi^{(1)'}(\rho_\varphi + P_\varphi) = aQ^{(0)}_\varphi\phi^{(1)} + a\delta^{(1)}Q_\varphi\,, \tag{4.3.37}$$
$$\delta^{(1)}\rho_\gamma + 3\mathcal{H}\left(\delta^{(1)}\rho_\gamma + \delta^{(1)}P_\gamma\right) - 3\psi^{(1)'}(\rho_\gamma + P_\gamma) = aQ^{(0)}_\gamma\phi^{(1)} + a\delta^{(1)}Q_\gamma \tag{4.3.38}$$

where we have dropped gradient terms, which are negligible on large scales.

Using the first order $(0-0)$ component of Einstein's equations on large scales $\psi^{(1)'} + \mathcal{H}\phi^{(1)} = -\frac{\mathcal{H}}{2}\frac{\delta^{(1)}\rho}{\rho}$, we can rewrite these equations in terms of the gauge-invariant curvature perturbations as

$$\zeta^{(1)'}{}_\varphi = \frac{a\mathcal{H}}{\rho'_\varphi}\left[\delta^{(1)}Q_\varphi - \frac{Q^{(0)'}_\varphi}{\rho'_\varphi}\delta^{(1)}\rho_\varphi + Q^{(0)}_\varphi\frac{\rho'}{2\rho}\left(\frac{\delta^{(1)}\rho_\varphi}{\rho_\varphi} - \frac{\delta^{(1)}\rho}{\rho}\right)\right]. \tag{4.3.39}$$
$$\zeta^{(1)'}{}_\gamma = \frac{a\mathcal{H}}{\rho'_\gamma}\left[\delta^{(1)}Q_\gamma - \frac{Q^{(0)'}_\gamma}{\rho'_\gamma}\delta^{(1)}\rho_\gamma + Q^{(0)}_\gamma\frac{\rho'}{2\rho}\left(\frac{\delta^{(1)}\rho_\gamma}{\rho_\gamma} - \frac{\delta^{(1)}\rho}{\rho}\right)\right]. \tag{4.3.40}$$

Now, we suppose that there is a perturbation in the decay rate

$$\Gamma(\tau, \mathbf{x}) = \Gamma(\tau) + \delta^{(1)}\Gamma(\tau, \mathbf{x}) \tag{4.3.41}$$

and so we write $\delta^{(1)}Q_\varphi$ as

$$\delta^{(1)}Q_\varphi = -\Gamma\delta^{(1)}\rho_\varphi - \delta^{(1)}\Gamma\rho_\varphi = -\delta^{(1)}Q_\gamma\,. \tag{4.3.42}$$

We take the background value $\Gamma(\tau) = \Gamma_*$ to be constant in time: in such a case $\delta^{(1)}\Gamma$ is a gauge-invariant quantity. Using the expression for $\delta^{(1)}Q_\varphi$ and the definition of the curvature perturbation, and using the cosmic time variable we obtain

$$\dot\zeta^{(1)}_\varphi = \frac{\Gamma}{2}\frac{\rho_\varphi}{\dot\rho_\varphi}\frac{\dot\rho}{\rho}\left(\zeta^{(1)} - \zeta^{(1)}_\varphi\right) + H\frac{\rho_\varphi}{\dot\rho_\varphi}\delta^{(1)}\Gamma\,. \tag{4.3.43}$$

We now adopt a "mixed sudden-decay approximation". We treat the two fluids as if they are not interacting until the decay of the inflaton when $\Gamma \simeq H$. Now, at the beginning of the reheating stage the energy density in radiation is negligible, that is $f = \frac{\dot\rho_\varphi}{\dot\rho} \simeq 1$ and $\zeta^{(1)} \simeq \zeta^{(1)}{}_\varphi$. In fact, under our approximation, we can neglect all the terms proportional to Γ, but we allow for the spatial fluctuations of the decay rate. So eq. (4.3.43) gives

$$\dot{\zeta}^{(1)}_{\varphi} \simeq -\frac{1}{3}\delta^{(1)}\Gamma \tag{4.3.44}$$

using the continuity equation $\dot{\rho}_{\varphi} = -3H\rho_{\varphi}$ in the sudden decay approximation.

Integrating over time we finally find

$$\zeta^{(1)}{}_{\varphi} = -\frac{t}{3}\delta^{(1)}\Gamma = -\frac{2}{9}\frac{\delta^{(1)}\Gamma}{H} \simeq \zeta^{(1)} \tag{4.3.45}$$

using the fact that during the oscillations of the scalar field $H = \frac{2}{3}t$.

The inhomogeneous reheating mechanism produces a gravitational potential which after the reheating phase is given by (see ref. (76))

$$\psi^{(1)} = \frac{1}{9}\frac{\delta^{(1)}\Gamma}{\Gamma_*}\,. \tag{4.3.46}$$

Using the relation in the radiation dominated era $\psi^{(1)} = -\frac{2}{3}\zeta^{(1)}$, we see from eq. (4.3.45) that we can assume $\frac{\Gamma_*}{H_D} = \frac{3}{4}$ in order to reproduce the numerical result (4.3.46). So finally we have the curvature perturbation at first order

$$\zeta^{(1)} \simeq -\frac{1}{6}\frac{\delta^{(1)}\Gamma}{\Gamma_*}\,. \tag{4.3.47}$$

Second order

We now expand the decay rate up to second order as

$$\Gamma = \Gamma_* + \delta^{(1)}\Gamma + \frac{1}{2}\delta^{(2)}\Gamma \tag{4.3.48}$$

so the perturbed energy transfer coefficient is

$$\delta^{(2)}Q_{\varphi} = -\rho_{\varphi}\delta^{(2)}\Gamma - \Gamma_*\delta^{(2)}\rho_{\varphi} - 2\delta^{(1)}\Gamma\delta^{(1)}\rho_{\varphi}\,. \tag{4.3.49}$$

Following the same steps as for the first order case, we obtain the equation of motion for the curvature perturbation $\zeta^{(2)}{}_{\varphi}$ on large scales:

$$\begin{aligned}\dot{\zeta}^{(2)}_{\varphi} =& \frac{H}{\dot{\rho}_{\varphi}}\left(\rho_{\varphi}\delta^{(2)}\Gamma + 2\delta^{(1)}\Gamma\delta^{(1)}\rho_{\varphi}\right) - \frac{\Gamma_*\rho_{\varphi}}{2}\frac{\dot{\rho}}{\rho}\left(\frac{\delta^{(2)}\rho_{\varphi}}{\dot{\rho}_{\varphi}} - \frac{\delta^{(2)}\rho}{\dot{\rho}}\right) + \\ &+ 3\Gamma_*\rho_{\varphi}\frac{H}{\dot{\rho}_{\varphi}}\left(\phi^{(1)}\right)^2 + \frac{2H}{\dot{\rho}_{\varphi}}\left(\delta^{(1)}\Gamma\rho_{\varphi} + \Gamma_*\delta^{(1)}\rho_{\varphi}\right)\phi^{(1)} - 2\dot{\zeta}^{(1)}_{\varphi}\zeta^{(1)}{}_{\varphi} + \\ &+ 2\left[\zeta^{(1)}{}_{\varphi}\left(\Gamma_*\frac{\rho_{\varphi}}{\dot{\rho}_{\varphi}}\phi^{(1)} + \delta^{(1)}\Gamma\frac{\rho_{\varphi}}{\dot{\rho}_{\varphi}} + \Gamma_*\frac{\delta^{(1)}\rho_{\varphi}}{\dot{\rho}_{\varphi}}\right)\right]^{\cdot} + \\ &+ \left[\frac{\Gamma}{H}\left(\zeta^{(1)}{}_{\varphi}\right)^2\left(1 - \frac{\rho_{\varphi}}{\dot{\rho}_{\varphi}}\frac{\dot{\rho}}{\rho}\right)\right]^{\cdot}\,.\end{aligned} \tag{4.3.50}$$

Under the sudden decay approximation and working in the spatially flat gauge we simplify this equation to

$$\dot{\zeta}^{(2)}_{\varphi} \simeq -\frac{1}{3}\delta^{(2)}\Gamma - \zeta^{(1)}{}_{\varphi}\delta^{(1)}\Gamma - 2\dot{\zeta}^{(1)}_{\varphi}\zeta^{(1)}{}_{\varphi} - \frac{2}{3}\left(\frac{\delta^{(1)}\Gamma}{H}\zeta^{(1)}{}_{\varphi}\right)^{\cdot} . \qquad (4.3.51)$$

Now, in this scenario the fluctuations of the decay rate depend on the underlying particle physics. Suppose for example that $\Gamma(t, \mathbf{x}) \propto \chi^2(t, \mathbf{x})$ where χ is a scalar field. If this field is light, the background can be treated as a constant, $\chi(t) \simeq \chi_*$, and small fluctuations $\delta^{(1)}\chi$ are left imprinted on superhorizon scales. In our example one has, up to second order

$$\Gamma(t, \mathbf{x}) \propto \chi^2(t, \mathbf{x}) = \chi_*^2 + 2\chi_*\delta^{(1)}\chi + \left(\delta^{(1)}\chi\right)^2 . \qquad (4.3.52)$$

From this equation we read

$$\frac{\delta^{(1)}\Gamma}{\Gamma_*} = 2\frac{\delta^{(1)}\chi}{\chi_*}, \qquad \frac{\delta^{(2)}\Gamma}{\Gamma_*} = 2\left(\frac{\delta^{(1)}\chi}{\chi_*}\right)^2 = \frac{1}{2}\left(\frac{\delta^{(1)}\Gamma}{\Gamma_*}\right)^2 \qquad (4.3.53)$$

and, using the first order solution, we rewrite the evolution equation as

$$\dot{\zeta}^{(2)}_{\varphi} \simeq -\frac{1}{6\Gamma_*}\left(\delta^{(1)}\Gamma\right)^2 + \frac{1}{3}\left(\delta^{(1)}\Gamma\right)^2 t - 2\dot{\zeta}^{(1)}_{\varphi}\zeta^{(1)}{}_{\varphi} - \frac{2}{3}\left(\frac{\delta^{(1)}\Gamma}{H}\zeta^{(1)}{}_{\varphi}\right)^{\cdot} . \qquad (4.3.54)$$

Integrating in time we get

$$\zeta^{(2)}{}_{\varphi} \simeq -\frac{t}{6\Gamma_*}\left(\delta^{(1)}\Gamma\right)^2 + \frac{1}{6}\left(\delta^{(1)}\Gamma\right)^2 t^2 - 2\left(\zeta^{(1)}{}_{\varphi}\right)^2 - \frac{2}{3}\frac{\delta^{(1)}\Gamma}{H}\zeta^{(1)}{}_{\varphi} . \qquad (4.3.55)$$

Now, at the time of inflaton decay $\frac{\Gamma_*}{H_D} = \frac{3}{4}$, and since $H = \frac{2}{3}t$, it follows $t_D = \frac{1}{2}\Gamma_*$. Evaluating $\zeta^{(2)}{}_{\varphi}$ at the time of inflaton decay we get

$$\zeta^{(2)}{}_{\varphi} \simeq -\frac{1}{24}\left(\frac{\delta^{(1)}\Gamma}{\Gamma_*}\right)^2 - \left(\zeta^{(1)}{}_{\varphi}\right)^2 - \frac{1}{2}\zeta^{(1)}{}_{\varphi}\frac{\delta^{(1)}\Gamma}{\Gamma_*} , \qquad (4.3.56)$$

and finally, using the first order result (4.3.47) we have

$$\zeta^{(2)} \simeq \zeta^{(2)}{}_{\varphi} \simeq \frac{1}{2}\left(\zeta^{(1)}{}_{\varphi}\right)^2 \qquad (4.3.57)$$

that is larger than the non-linearity produced in the standard scenario.

4.4 Evolution of second order perturbations

We have shown that, after it has been generated during inflation, the gauge invariant curvature perturbation remains constant on large scales, for adiabatic perturbations.

We now want to study the evolution of non-linearities in the gravitational potentials during the radiation and matter dominated epochs [32]; from now on we shall adopt the Poisson gauge, defined by $\omega^{(r)} = \chi^{(r)} = \chi_i^{(r)} = 0$. This way we have two scalars, $\phi^{(r)}$ and $\psi^{(r)}$, two vector degrees of freedom $\omega_i^{(r)}$ and two tensor degrees of freedom $\chi_{ij}^{(r)}$.

We consider the second order energy continuity equation at second order for a perfect fluid with generic equation of state $P = w\rho$, $w = \text{const}$:

$$\begin{aligned}
&\delta^{(2)}\rho' + 3\mathcal{H}(1+w)\delta^{(2)}\rho - 3(1+w)\rho_0\psi^{(2)'} - 6(1+w)\psi^{(1)'}\left[\delta^{(1)}\rho + 2\rho_0\psi^{(1)}\right] = \\
&= -2(1+w)\rho_0\left(v_i^{(1)}v_{(1)}^i\right)' - 2(1+w)(1-3w)\mathcal{H}\rho_0 v_i^{(1)}v_{(1)}^i + \\
&\quad + 4(1+w)\rho_0 v_{(1)}^i\partial_i\psi^{(1)} + 2\frac{\rho_0}{\mathcal{H}^2}\left(\psi^{(1)}\nabla^2\psi^{(1)'} - \psi^{(1)'}\nabla^2\psi^{(1)}\right) ,
\end{aligned} \tag{4.4.1}$$

where we have also used the divergence of the $(0-i)$ second order Einstein equation.

This equation can be rewritten in a more suitable form

$$\begin{aligned}
&\left[\psi^{(2)} + \mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'} + (1+3w)\mathcal{H}^2\left(\frac{\delta^{1)}\rho}{\rho_0'}\right)^2 - 4\mathcal{H}\left(\frac{\delta^{1)}\rho}{\rho_0'}\right)\psi^{(1)}\right]' = \\
&= \frac{2}{3}\left(v_i^{(1)}v_i^{(1)}\right)' + \frac{2}{3}(1-3w)\mathcal{H}v_i^{(1)}v_{(1)}^i - \frac{4}{3}v_{(1)}^i\partial_i\psi^{(1)} + \\
&\quad + \frac{16}{27\mathcal{H}(1+w)^2}\psi^{(1)}\nabla^2\psi^{(1)} - \frac{2}{3\mathcal{H}^2(1+w)}\times \\
&\quad \times\left\{\left[1 - \frac{8}{9(1+w)}\right]\psi^{(1)}\nabla^2\psi^{(1)'} - \left[1 - \frac{4(1+3w)}{9(1+w)}\right]\psi^{(1)'}\nabla^2\psi^{(1)}\right\} + \\
&\quad + \frac{8(1+3w}{27\mathcal{H}^3(1+w)^2}\left[\frac{1}{3}\left(\nabla^2\psi^{(1)}\right)^2 - \psi^{(1)'}\nabla^2\psi^{(1)'} + \frac{1}{3\mathcal{H}}\nabla^2\psi^{(1)'}\nabla^2\psi^{(1)}\right]
\end{aligned} \tag{4.4.2}$$

where we have employed the $(0-0)$ first order equation

$$6\mathcal{H}^2\phi^{(1)} + 6\mathcal{H}\psi^{(1)'} - 2\nabla^2\psi^{(1)} = -8\pi G a^2\delta^{(1)}\rho \tag{4.4.3}$$

and the constraint from the traceless part of the $(i-j)$ equation

$$\phi^{(1)} = \psi^{(1)} \tag{4.4.4}$$

to get, on large scales, the relation

$$\psi^{(1)} = -\frac{1}{2}\frac{\delta^{(1)}\rho}{\rho_0} = \frac{3}{2}\mathcal{H}(1+w)\frac{\delta^{(1)}\rho}{\rho_0'} \,. \tag{4.4.5}$$

Using the definition of $\zeta^{(1)}$ and the last equation, we find

$$\psi^{(1)} = -\frac{3(1+w)}{5+3w}\zeta^{(1)} \tag{4.4.6}$$

which relates the gravitational potential on large scales to the conserved curvature perturbation.

The LHS of our evolution equation can be further simplified to

$$\begin{aligned}
&\psi^{(2)} + \mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'} + (1+3w)\mathcal{H}^2\left(\frac{\delta^{1)}\rho}{\rho_0'}\right)^2 - 4\mathcal{H}\left(\frac{\delta^{1)}\rho}{\rho_0'}\right)\psi^{(1)} = \\
=&\psi^{(2)} + \mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'} + (5+3w)\mathcal{H}^2\left(\frac{\delta^{1)}\rho}{\rho_0'}\right)^2 = \\
=&\psi^{(2)} + \mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'} - \frac{4}{5+3w}\left(\zeta_I^{(1)}\right)^2
\end{aligned} \tag{4.4.7}$$

where $\zeta_I^{(1)}$ is the curvature perturbation evaluated at the end of inflation.

In fact, this expression can be recognized as the second-order curvature perturbation in the case of a generic fluid with constant equation of state:

$$\begin{aligned}
-\zeta^{(2)} =&\hat{\psi}^{(2)} + \mathcal{H}\frac{\delta^{(2)}\rho}{\rho'} - 2\mathcal{H}\frac{\delta^{(1)}\rho'}{\rho'}\frac{\delta^{(1)}\rho}{\rho'} - 2\frac{\delta^{(1)}\rho}{\rho'}\left(\hat{\psi}^{(1)\prime} + 2\mathcal{H}\hat{\psi}^{(1)}\right) + \\
&+ \left(\frac{\delta^{(1)}\rho}{\rho'}\right)^2\left(\mathcal{H}\frac{\rho''}{\rho} - \mathcal{H}' - 2\mathcal{H}^2\right) = \\
=&\psi^{(2)} + \mathcal{H}\frac{\delta^{(2)}\rho}{\rho'} - (1+3w)\mathcal{H}\left(\frac{\delta^{(1)}\rho}{\rho'}\right)^2 + 4\mathcal{H}\left(\frac{\delta^{(1)}\rho}{\rho'}\right)\psi^{(1)} \,.
\end{aligned} \tag{4.4.8}$$

So we have

$$\psi^{(2)} + \mathcal{H}\frac{\delta^{(2)}\rho}{\rho_0'} - \frac{4}{5+3w}\left(\zeta_I^{(1)}\right)^2 = \mathcal{C} + \frac{2}{3}v_i^{(1)}v_{(i)}^i + \int^\tau d\tau'\mathcal{S}(\tau') \tag{4.4.9}$$

where $\mathcal{S}$ is the source term, and on large scales can be neglected together with the velocities; $\mathcal{C}$ is a constant, and in fact it is the constant value of the curvature perturbation produced during inflation.

Finally, we can find a relation between $\psi^{(2)}$ and the density perturbation at second order:

$$\psi^{(2)} - \frac{1}{3(1+w)}\frac{\delta^{(2)}\rho}{\rho_0} = -\frac{2}{3}\frac{5+3w}{1+w}\left(\psi^{(1)}\right)^2 + \frac{2}{3}v_i^{(1)}v^i_{(i)} + \int^\tau d\tau'\mathcal{S}(\tau')\,. \tag{4.4.10}$$

Now, from this equation, the second order $(0-0)$ Einstein equation

$$\begin{aligned}&3\mathcal{H}^2\phi^{(2)} + 3\mathcal{H}\psi^{(2)\prime} - \nabla^2\psi^{(2)} - 12\mathcal{H}^2\left(\psi^{(1)}\right)^2 - 3\left(\nabla^2\psi^{(1)}\right)^2 + \\ &- 8\psi^{(1)}\nabla^2\psi^{(1)} - 3\left(\psi^{(1)\prime}\right)^2 = 8\pi G a^2\delta^{(2)}T^0_0\end{aligned} \tag{4.4.11}$$

and the relation

$$\begin{aligned}\psi^{(2)} - \phi^{(2)} = \mathcal{S}_1 = &- 4\left(\psi^{(1)}\right)^2 - \nabla^{-2}\left[2\partial^i\psi^{(1)}\partial_i\psi^{(1)} + 3(1+w)\mathcal{H}^2 v_i^{(1)}v^i_{(i)}\right] + \\ &+ 3\nabla^{-4}\partial_i\partial^j\left[2\partial^i\psi^{(1)}\partial_j\psi^{(1)} + 3(1+w)\mathcal{H}^2 v_i^{(1)}v^j_{(i)}\right]\end{aligned} \tag{4.4.12}$$

we find an evolution equation for the second order potential $\phi^{(2)}$:

$$\begin{aligned}\phi^{(2)\prime} + \frac{5+3w}{2}\mathcal{H}\phi^{(2)} = &(5+3w)\mathcal{H}\left(\psi^{(1)}\right)^2 + \\ &+ \frac{3}{2}\mathcal{H}(1+w)\left\{\nabla^{-2}\left[2\partial^i\psi^{(1)}\partial_i\psi^{(1)} + 3(1+w)\mathcal{H}^2 v_i^{(1)}v^i_{(i)}\right] + \right.\\ &\left. - 3\nabla^{-4}\partial_i\partial^j\left[2\partial^i\psi^{(1)}\partial_j\psi^{(1)} + 3(1+w)\mathcal{H}^2 v_i^{(1)}v^j_{(i)}\right]\right\} + \\ &+ \frac{3}{2}\mathcal{H}(1+w)\int_{\tau_I}^{\tau}\mathcal{S}(\tau')d\tau' - \mathcal{S}_1' + \frac{1}{\mathcal{H}}\left(\nabla\psi^{(1)}\right)^2 + \\ &+ \frac{8}{3\mathcal{H}}\psi^{(1)}\left(\nabla\psi^{(1)}\right) + \frac{1}{3\mathcal{H}}\nabla^2\mathcal{S}_1 + \frac{1}{\mathcal{H}}\left(\psi^{(1)\prime}\right)^2\,.\end{aligned} \tag{4.4.13}$$

We want to integrate this equation from the end of inflation τ_I to a time τ in the matter dominated era (we have in mind the epoch of recombination). The general solution is given by the solution of the homogeneous equation plus a particular solution:

$$\begin{aligned}\phi^{(2)} = &\phi^{(2)}(\tau_I)\exp\left[-\int_{\tau_I}^{\tau}\frac{5+3w}{2}\mathcal{H}d\tau'\right] \\ &+ \exp\left[-\int_{\tau_I}^{\tau}\frac{5+3w}{2}\mathcal{H}d\tau'\right]\times\int_{\tau_I}^{\tau}\exp\left[-\int_{\tau_I}^{\tau'}\frac{5+3w}{2}\mathcal{H}ds\right]b(\tau')d\tau'\end{aligned} \tag{4.4.14}$$

where $b(\tau)$ is the source term of eq. (4.4.13).

The homogeneous solution decreases in time both during the radiation and matter dominated epochs, so we can neglect it. Since at a time τ during matter domination $\exp\left[-\int_{\tau_I}^{\tau}\frac{5+3w}{2}\mathcal{H}d\tau'\right]\propto\tau^{-5}$, we can neglect the contributions from the radiation-dominated epoch. Recalling that during the matter-dominated epoch $\psi^{(1)}$ is constant in time, it turns out that

$$\phi^{(2)}\simeq 2\left(\psi^{(1)}\right)^2+\frac{3}{5}\left[\nabla^{-2}\left(\frac{10}{3}\partial^i\psi^{(1)}\partial_i\psi^{(1)}\right)-3\nabla^{-4}\partial_i\partial^j\left(\frac{10}{3}\partial^i\psi^{(1)}\partial_j\psi^{(1)}\right)\right]+$$
$$+\exp\left[-\int_{\tau_I}^{\tau}\frac{5+3w}{2}\mathcal{H}d\tau'\right]\int_{\tau_I}^{\tau}\exp\left[-\int_{\tau_I}^{\tau'}\frac{5+3w}{2}\mathcal{H}ds\right]\left\{\frac{3}{2}\mathcal{H}(1+w)\times\right.$$
$$\left.\times\int_{\tau_I}^{\tau'}\mathcal{S}(s)ds+\frac{1}{\mathcal{H}}\left(\nabla^2\psi^{(1)}\right)^2+\frac{8}{3\mathcal{H}}\psi^{(1)}\left(\nabla^2\psi^{(1)}\right)+\frac{1}{3\mathcal{H}}\nabla^2\mathcal{S}_1\right\}d\tau'\,. \tag{4.4.15}$$

where we have used the $(0-i)$ first order equation to express the spatial velocities in terms of $\psi^{(1)}$, and we have taken into account that during the matter dominated era $\mathcal{S}_1'=0$.

The gravitational potential will then have a non-Gaussian (χ^2)-component. Going to momentum space, we directly read the non-linearity parameter for scales entering the horizon during the matter-dominated era:

$$f_{NL}^{\phi}\simeq-\frac{1}{2}+g(\mathbf{k}_1,\mathbf{k}_2) \tag{4.4.16}$$

where

$$g(\mathbf{k}_1,\mathbf{k}_2)=4\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k^2}-3\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k^4}+\frac{3}{2}\frac{k_1^4+k_2^4}{k^4}\,, \tag{4.4.17}$$

with $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2$.

In deriving this result, we neglect the last term in equation (4.4.15), because it is fully negligible when evaluationg the bispectrum of the gravitational potential on large scales.

We note also that in the final bisectrum expression the diverging terms arising from the infrared behaviour of $f_{NL}(\mathbf{k}_1,\mathbf{k}_2)$ are automatically regularized once we subtract the monopole term, by requiring $\langle\phi\rangle=0$.

So we conclude that in the standard scenario the tiny non-Gaussianity produced during inflation gets enhanced during the post inflationary evolution, giving rise to a non-negligible signature of large-scale non-Gaussianity in the gravitational potentials.

4.5 Sachs-Wolfe effect

At this point, we have to discuss how metric perturbations are converted into temperature fluctuations on large scales at the recombination epoch.

Calculating the photon redshift up to second order, and dropping for the moment integrated effects, we obtain the non-linear generalization of the Sachs-Wolfe effect [33, 34]

$$\frac{\delta T}{T} = \phi_{\mathcal{E}}^{(1)} + \tau_{\mathcal{E}}^{(1)} + \frac{1}{2}\left(\phi_{\mathcal{E}}^{(2)} + \tau_{\mathcal{E}}^{(2)}\right) - \frac{1}{2}\left(\phi_{\mathcal{E}}^{(1)}\right)^2 + \phi_{\mathcal{E}}^{(1)}\tau_{\mathcal{E}}^{(1)} \tag{4.5.1}$$

where $\phi_{\mathcal{E}} = \phi_{\mathcal{E}}^{(1)} + \frac{1}{2}\phi_{\mathcal{E}}^{(2)}$ is the lapse perturbation at emission on the last scattering surface and $\tau_{\mathcal{E}} = \tau_{\mathcal{E}}^{(1)} + \frac{1}{2}\tau_{\mathcal{E}}^{(2)}$ is the intrinsic fractional temperature fluctuation at emission.

To obtain the intrinsic anisotropy in the photon temperature, we expand the relation $\rho_\gamma \propto T^4$ up to second order:

$$\tau_{\mathcal{E}}^{(1)} = \frac{1}{4}\left.\frac{\delta^{(1)}\rho_\gamma}{\rho_\gamma}\right|_{\mathcal{E}} \tag{4.5.2}$$

$$\tau_{\mathcal{E}}^{(2)} = \frac{1}{4}\left.\frac{\delta^{(2)}\rho_\gamma}{\rho_\gamma}\right|_{\mathcal{E}} - 3\left(\tau_{\mathcal{E}}^{(1)}\right)^2 = \frac{1}{4}\left.\frac{\delta^{(2)}\rho_\gamma}{\rho_\gamma}\right|_{\mathcal{E}} - \frac{3}{16}\left(\left.\frac{\delta^{(1)}\rho_\gamma}{\rho_\gamma}\right|_{\mathcal{E}}\right)^2 \tag{4.5.3}$$

where ρ_γ is the mean photon energy density.

Then we need to relate the photon energy density to the lapse perturbation, which we can do easily by implementing the adiabaticity condition. At first order $\zeta_m^{(1)} = \zeta_\gamma^{(1)}$ and we find

$$\frac{\delta^{(1)}\rho_\gamma}{\rho_\gamma} = \frac{4}{3}\frac{\delta^{(1)}\rho_m}{\rho_m}\,; \tag{4.5.4}$$

at second order we have $\zeta_m^{(2)} = \zeta_\gamma^{(2)}$ and using the definition of $\zeta^{(2)}$ we obtain

$$\frac{\delta^{(2)}\rho_\gamma}{\rho_\gamma} = \frac{4}{3}\frac{\delta^{(2)}\rho_m}{\rho_m} + \frac{4}{9}\left(\frac{\delta^{(1)}\rho_m}{\rho_m}\right)^2 . \tag{4.5.5}$$

In the large scale limit, the energy constraints yield the solutions

$$\frac{\delta^{(1)}\rho_m}{\rho_m} = -2\psi^{(1)} \tag{4.5.6}$$

$$\frac{\delta^{(2)}\rho_m}{\rho_m} = -2\phi^{(2)} + 8\left(\psi^{(1)}\right)^2 . \tag{4.5.7}$$

Finally, we obtain the second-order generalization of the Sachs-Wolfe result

$$\frac{\delta T}{T} = \frac{1}{3}\left[\psi_{\mathcal{E}}^{(1)} + \frac{1}{2}\left(\phi_{\mathcal{E}}^{(2)} - \frac{5}{3}\left(\psi^{(1)}\right)^2\right)\right] \tag{4.5.8}$$

which is not a simple extension of the formula $\frac{\delta^{(1)}T}{T} = \frac{1}{3}\psi_{\mathcal{E}}^{(1)}$ since it receives a correction provided by the term $-\frac{5}{3}\left(\psi^{(1)}\right)^2$.

Expressing the lapse function at second order as a general convolution, and recalling that at linear order $\phi^{(1)} = \psi^{(1)}$,

$$\phi = \phi^{(1)} + \frac{1}{2}\phi^{(2)} = \psi^{(1)} + f^{\phi}_{NL} \star \left(\psi^{(1)}\right)^2 \tag{4.5.9}$$

up to a constant offset that must be added in order to have $\langle\phi\rangle = 0$.

In order to connect the inflationary predictions with the definition of f_{NL} which has become standard in the CMB-related literature we remind the standard Sachs-Wolfe formula

$$\frac{\delta T}{T}(\hat{\mathbf{n}}, \tau_0) = -\frac{1}{3}\Phi(\hat{\mathbf{n}}(\tau_0 - \tau_{\mathcal{E}}))\,, \tag{4.5.10}$$

where $\Phi = -\phi$ is Bardeen's gauge-invariant potential, which is usually expanded as

$$\Phi = \Phi_L + f_{NL} \star (\Phi_L)^2 \tag{4.5.11}$$

(up to a constant offset), where $\Phi_L = -\phi^{(1)}$.

Finally, by comparison we immediately derive the true non linearity parameter which is measured by CMB experiments:

$$f_{NL} = -f^{\phi}_{NL} + \frac{5}{6}\,. \tag{4.5.12}$$

We notice that the CMB temperature bispectrum does not vanish even if $f^{\phi}_{NL} \simeq 0$, owing to the presence of the second-order Sachs-Wolfe effect that give the extra term $\frac{5}{6}$.

In deriving second order quantities and equations of motion, we have sistematically dropped spatial gradients since they are negligible on large scales. However, second order perturbations are sourced by first order ones, and we must take into account also the short wavelength behaviour of the first order quantities, as it is evident going to momentum space.

The point here is the final quantity one is interested in. We want to calculate the bispectrum of the gravitational potentials and of the temperature anisotropies as a measure of non-Gaussianity on large scales. Now, by eq. (4.1.3) we see that the bispectrum is given by the kernel f_{NL} which appears when expressing in Fourier space second order quantities in terms of first order ones, so when calculating the bispectrum on large scales we need only to evaluate the kernel in the long-wavelength limit, irrespective of the integration over the whole range of momenta.

This is the reason why we can use the large-scale limit when deriving and solving equations for the second order quantities in terms of first order ones: this procedure does not affect the value of f_{NL} on large scales and so the final result for the bispectrum is correct.

Chapter 5

Second order radiation transfer function on large scales

In this Chapter, we want to compute the second order temperature anisotropies including all the relevant effects on large scales. We start by discussing the integrated Sachs-Wolfe effect, which vanishes in the case of a matter dominated universe. Then we present a non-perturbative formalism to obtain an expression for the temperature anisotropies and the evolution equations on large scales. Neglecting integrated effects, we can use these formalism to compute at all order in perturbation theory the bispectrum and trispectrum. In the general case, we can perturb our equations to study all the relevant integrated effects on large scales for the temperature anisotropies, namely the early and late ISW, and the second-order tensor contribution.

5.1 Integrated Sachs-Wolfe effect

After recombination, CMB photons essentially free-stream to us. During their path from the last scattering surface to the observer, there are sources of anisotropies, in addition to those left imprinted on the CMB at the moment of recombination.

These effects are purely of gravitational origin, and are of two types: frequency change due to gravitational redshift along the photon path, and gravitational lensing which modifies the angular distribution of anisotropies.

Second order anisotropies could be important, because they can give a non negligible contribution with respect to the first order ones due to the long distances involved in the problem, since some second order terms consist of integrals along the photon path. Clearly, we are interested in the second order integrated effect because we want to take into account all the contributions to the second-order temperature anisotropies to give a definite prediction for the observed level of non-Gaussianity n the CMB on large scales.

On large scales, the most important effect is the so-called integrated Sachs-Wolfe effect (ISW), which is due to differential redshift and time dilation integrated along the photon path. The linear ISW effect vanishes in a matter dominated universe, since the gravitational potentials are constant in time and their time derivatives are zero: at second order, however, the integrated effect is present and is often referred to as the Rees-Sciama effect [35], especially on subhorizon scales.

We can distinguish two different effects: the early ISW effect, at recombination epoch, which is due to a residual component of radiation that makes the potential decay in time; and the late ISW effect, which is present at late times and is due to the dark energy component.

At linear order, the expression for the ISW effect was obtained by Sachs and Wolfe considering the redshift experienced by a photon along its geodesic.

A perturbative algorithm to compute second and higher order gravitational perturbations on the CMB is developed in ref. [36] (see also [37]): the explicit computation up to second order for a perturbed FRW model was performed in ref. [34].

To understand the origin of the integrated effects, we present the explicit computation at first order in the perturbations [33].

Let us consider a flat FRW spacetime perturbed up to first order: the line element is

$$ds^2 = a^2(\eta)\left[g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu}\right] dx^\mu dx^\nu \tag{5.1.1}$$

and the metric components are given by eq. (3.3.25). The perturbations to g_{0i} will be denoted by z instead of ω to not generate confusion, and for simplicity of notation in the following we will drop the superscript $^{(1)}$.

Photons travel on null geodesics $x^\mu(\lambda)$, with λ the affine parameter in the conformal metric, connecting the observer, at coordinates $\mathbf{x}^\mu_{\mathcal{O}} = (\eta_{\mathcal{O}}, \mathbf{x}_{\mathcal{O}})$, to the emitting hypersurface, that we take at constant conformal time $\eta_{\mathcal{E}}$, and that can assumed to be the last scattering surface.

At every point p_i on this hypersurface is emitted thermal radiation with temperature $T_{\mathcal{E}}(p_i, d_i)$ where d_i is a unit vector specifying the direction of emission.

At our observation point, we measure a temperature $T_{\mathcal{O}}(\mathbf{x}_O, e_i)$, where e_i is the direction of observation.

If the photon suffers a redshift z along its travel, the frequencies at emission and at observation will be related by $\omega_{\mathcal{O}} = \frac{\omega_{\mathcal{E}}}{1+z}$; since the occupation number per frequency mode is conserved, the temperatures will be related by $T_{\mathcal{O}} = \frac{T_{\mathcal{E}}}{1+z}$.

So, the temperature measured by an observer can be written as

$$T_{\mathcal{O}}(x_i, e_i) = \frac{\omega_{\mathcal{O}}}{\omega_{\mathcal{E}}} T_{\mathcal{E}}(p_i, d_i)\,. \tag{5.1.2}$$

The frequency is given by

$$\omega = -g_{\mu\nu} u^\mu k^\nu \tag{5.1.3}$$

where u^μ is the four-velocity of the observer or the emitter, and $k^\nu = \frac{dx^\nu}{d\lambda}$ is the wavevector of the photon in the conformal metric, tangent to the null geodesic $x^\nu(\lambda)$.

As initial data we have the quantities $\mathbf{x}_\mathcal{O}$, e_i and $\omega_\mathcal{O}$, and we need to obtain $\omega_\mathcal{E}$, p_i, d_i. We expand perturbatively the photon geodesics and the corresponding wavevectors as

$$x^\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda) + \dots \tag{5.1.4}$$

$$k^\mu(\lambda) = k^{(0)\mu}(\lambda) + k^{(1)\mu}(\lambda) + \dots\,. \tag{5.1.5}$$

For simplicity, we can take comoving observers at $\mathcal{O}$ and $\mathcal{E}$, because any relative motion leads to a dipole anisotropy which may be easily subtracted. Therefore our four-velocity will be

$$u^\mu = \frac{1}{a}\left(1 + v^{(1)0}, 0, 0, 0\right) \tag{5.1.6}$$

and the normalization condition $g_{\mu\nu} u^\mu u^\nu = -1$ gives

$$v^{(1)0} = -\phi\,. \tag{5.1.7}$$

Finally, we expand the frequency as

$$\omega = \omega_0\,(1 + \tilde{\omega}) \tag{5.1.8}$$

and the temperature at emission as

$$T_\mathcal{E}(p_i, d_i) = T_\mathcal{E}^{(0)}\,(1 + \tau(p_i, d_i)) \tag{5.1.9}$$

where τ is the intrinsic temperature at the emission point.

So, up to first order, the temperature at the observer is

$$T_\mathcal{O}(\mathbf{x}_\mathcal{O}, e_i) = \frac{\omega_\mathcal{O}^{(0)}}{\omega_\mathcal{E}^{(0)}} T_\mathcal{E}^{(0)}\,[1 + (\tilde{\omega}_\mathcal{O} - \tilde{\omega}_\mathcal{E} + \tau)] \tag{5.1.10}$$

where the first factor gives the mean temperature at observation $T_\mathcal{O}^{(0)} \equiv \frac{\omega_\mathcal{O}^{(0)}}{\omega_\mathcal{E}^{(0)}} T_\mathcal{E}^{(0)}$.

Now we have to write an expression for $T_\mathcal{O}$ in terms of null geodesics. As background geodesics, we use simply straight lines:

$$x^{(0)\mu} = (\lambda, (\lambda_\mathcal{O} - \lambda) e_i) \qquad k^{(0)\mu} = (1, -e_i)\;. \tag{5.1.11}$$

As boundary conditions at the origin, we impose $x^{(1)\mu}(\lambda_O) = k^{(1)i}(\lambda_O) = 0$. The null geodesic condition then gives

$$k^{(1)0}(\lambda_O) = -\phi_{\mathcal{O}} - z^i_{\mathcal{O}} e_i - \psi_{\mathcal{O}} + \frac{1}{2}\chi^{ij}_{\mathcal{O}} e_i e_j \,. \tag{5.1.12}$$

The expansion of ω gives

$$\omega^{(0)} = \frac{1}{a}\,, \quad \tilde{\omega} = k^{(1)0} + \phi^{(1)} + z^i e_i \tag{5.1.13}$$

and substituting into the expression for the temperature we find

$$\frac{\delta T}{T} = \tilde{\omega}_{\mathcal{O}} - \tilde{\omega}_{\mathcal{E}} + \tau = \psi_{\mathcal{O}} + \frac{1}{2}\chi^{ij}_{\mathcal{O}} e_i e_j - k^{(1)0}_{\mathcal{E}} - z^i_{\mathcal{E}} e_i - \psi_{\mathcal{E}} + \tau \,. \tag{5.1.14}$$

Now we need to obtain the null geodesics up to first order. The geodesic equation is

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} k^\alpha k^\beta = 0\,, \tag{5.1.15}$$

along the path $x^\mu(\lambda)$.

We seek an equivalent equation for the perturbations to the background geodesics. Substituting the expansion (3.3.27) we have

$$\sum_{a=0}^{\infty}\left\{\frac{d^2 x^{(a)\mu}}{d\lambda^2} + \left[\Gamma^{(a)\mu}_{\alpha\beta} + \sum_{b=1}^{\infty}\frac{1}{b!}\partial_{\sigma_1}\ldots\partial_{\sigma_b}\Gamma^{(a)\mu}_{\alpha\beta}\left(\sum_{c=1}^{\infty}x^{(c)\sigma_1}\right)\ldots\left(\sum_{d=1}^{\infty}x^{(d)\sigma_b}\right)\times \right.\right.$$
$$\left.\left. \times\left(\sum_{e=1}^{\infty}k^{(e)\alpha}\right)\left(\sum_{f=1}^{\infty}k^{(f)\beta}\right)\right]\right\} = 0 \tag{5.1.16}$$

which is an equivalent equation, holding along the background path $x^{(0)\mu}(\lambda)$, which can be solved perturbatively for the $x^{(a)\mu}(\lambda)$.

At every order the equation can be recasted in the form of a forced Jacobi equation:

$$\frac{d^2 x^{(a)\mu}}{d\lambda^2} + 2\Gamma^{(0)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(a)\beta} + \partial_\sigma \Gamma^{(0)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta} x^{(a)\sigma} = f^{(a)\mu} \tag{5.1.17}$$

where $f^{(a)\mu}$ is a forcing vector which contains $x^{(b)\mu}$ or $k^{(b)\mu}$ only up to the $(a-1)$-th order.

At first order, it is given by

$$f^{(1)\mu} = -\Gamma^{(1)\mu}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta}\,. \tag{5.1.18}$$

There is a general method for solving these equations, in terms of parallel and Jacobi propagators.

In our case, we find

$$k^{(1)0}(\lambda_{\mathcal{E}}) = \phi_{\mathcal{O}} - \psi_{\mathcal{O}} + \frac{1}{2}\chi^{ij}_{\mathcal{O}} e_i e_j - 2\phi_{\mathcal{E}} - z^i_{\mathcal{E}} e_i + \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} d\lambda A^{(1)\prime} \tag{5.1.19}$$

where $A^{(1)} \equiv \phi^{(1)} + \psi^{(1)} + z^{(1)}_i e^i - \frac{1}{2}\chi^{(1)}_{ij} e^i e^j$.

Substituting this expression into eq. (5.1.14), we finally find

$$\frac{\delta^{(1)} T}{T} = \phi^{(1)}_{\mathcal{E}} - \phi^{(1)}_{O} + \tau - \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} d\lambda A^{(1)\prime} \tag{5.1.20}$$

which is the generalization of the Sachs-Wolfe result.

The intrinsic temperature anisotropy τ obviously depend on the cosmological model in consideration. In the case of adiabatic perturbations, it is simply given by $\tau = -\frac{2}{3}\phi^{(1)}$.

5.2 Non-linear anisotropies

We now derive an expression for the temperature anisotropies on large scales using the following metric [38]

$$ds^2 = a^2 \left[-e^{2\Phi} d\eta^2 + e^{-2\Psi} \delta_{ij} dx^i dx^j \right] \tag{5.2.1}$$

where $a(\eta)$ is the scale factor of the unperturbed universe, and we have introduced the gravitational potentials Φ and Ψ in order to describe scalar perturbations (in the Poisson gauge) in a fully non-linear way. That is, expression (5.2.1) holds at any order in perturbation theory, and to make contact with the standard perturbative approach one has to expand the gravitational potentials as $\Phi = \sum_{n=1}^{\infty} \frac{1}{n!}\Phi_n$, $\Psi = \sum_{n=1}^{\infty} \frac{1}{n!}\Psi_n$.

For example, up to second order in perturbations we find

$$e^{2\Phi} = 1 + 2\Phi + 2\Phi^2 + O(\Phi^3) = 1 + 2\left(\Phi_1 + \frac{1}{2}\Phi_2\right) + 2\Phi_1^2 + O(\Phi_1^3)\,, \tag{5.2.2}$$

$$e^{-2\Psi} = 1 - 2\Psi + 2\Psi^2 + O(\Phi^3) = 1 - 2\left(\Psi_1 + \frac{1}{2}\Psi_2\right) + 2\Psi_1^2 + O(\Psi_1^3)\,, \tag{5.2.3}$$

and by comparing the usual perturbed metric in the Poisson gauge we find

$$\Phi_i = \phi^{(1)}\,, \quad \Psi_1 = \psi^{(1)}\,, \quad \Phi_2 = \phi^{(2)} - 2(\phi^{(1)})^2\,, \quad \Psi_2 = \psi^{(2)} + 2(\psi^{(1)})^2\,. \tag{5.2.4}$$

Since we are interested in the large scale limit, when using the metric (5.2.1) we neglect spatial gradients of the gravitational potentials.

Now we obtain a general non linear formula for the temperature anisotropies on large scales. The temperature measured by an observer is given by

$$T_{\mathcal{O}} = \frac{\omega_{\mathcal{O}}}{\omega_{\mathcal{E}}} T_{\mathcal{E}} \,. \tag{5.2.5}$$

The frequency seen by an observer with four-velocity U^μ is

$$\omega = -\frac{1}{a^2} g_{\mu\nu} U^\mu k^\nu \tag{5.2.6}$$

where $g_{\mu\nu}$ is the metric of eq. (5.2.1), and k^ν is the wavevector in the conformal metric.

For simplicity we take comoving observers in $\mathcal{O}$ and $\mathcal{E}$ (losing a dipole contribution), so we write $U^\mu = (U^0, \vec{0})$. The normalization condition $g_{\mu\nu}U^\mu U^\nu = -1$ gives $U^0 = \frac{1}{a}e^{-\Phi}$, and substituting in eq. (5.2.6) we find

$$\omega = \frac{1}{a} e^{\Phi} k^0 \,. \tag{5.2.7}$$

which expanded up to second order reproduces the results in ref. Mollerach-Matarrese.

The non-linear expression for the temperature anisotropy is

$$T_{\mathcal{O}} = T_{\mathcal{E}} \frac{a_{\mathcal{E}}}{a_{\mathcal{O}}} e^{\Phi_{\mathcal{O}} - \Phi_{\mathcal{E}}} \frac{k^0_{\mathcal{O}}}{k^0_{\mathcal{E}}} \,. \tag{5.2.8}$$

This expression is valid at any order in perturbation theory. To recover the second order result, we simply expand the terms as

$$T_{\mathcal{E}} \simeq T^{(0)}_{\mathcal{E}} \left(1 + \tau^{(1)} + \tau^{(2)}\right) \tag{5.2.9}$$

$$e^{\Phi_{\mathcal{O}} - \Phi_{\mathcal{E}}} \simeq 1 + \Phi_{1\mathcal{O}} - \Phi_{1\mathcal{E}} + \frac{1}{2}\left(\Phi_{2\mathcal{O}} - \Phi_{2\mathcal{E}}\right) + \frac{1}{2}\left(\Phi_{1\mathcal{O}} - \Phi_{1\mathcal{E}}\right)^2 \tag{5.2.10}$$

$$= 1 + \phi^{(1)}\Big|_{\mathcal{E}}^{\mathcal{O}} + \frac{1}{2}\,\phi^{(2)}\Big|_{\mathcal{E}}^{\mathcal{O}} - \frac{1}{2}\left(\phi^{(1)}_{\mathcal{O}}\right)^2 + \frac{3}{2}\left(\phi^{(1)}_{\mathcal{E}}\right)^2 - \phi^{(1)}_{\mathcal{O}}\phi^{(1)}_{\mathcal{E}} \tag{5.2.11}$$

$$k^0_{\mathcal{O}} \simeq 1 + k^{(1)0}_{\mathcal{O}} + k^{(2)0}_{\mathcal{O}} \tag{5.2.12}$$

$$\frac{1}{k^0_{\mathcal{E}}} \simeq 1 - k^0_{\mathcal{E}} + \left(k^0_{\mathcal{E}}\right)^2 = 1 - k^{(1)0}_{\mathcal{E}} - k^{(2)0}_{\mathcal{E}} + \left(k^{(1)0}_{\mathcal{E}}\right)^2 \,. \tag{5.2.13}$$

At first order we obtain

$$\frac{\delta T^{(1)}}{T} = \tau^{(1)} + \phi^{(1)}_{\mathcal{O}} - \phi^{(1)}_{\mathcal{E}} + k^{(1)0}_{\mathcal{O}} - k^{(1)0}_{\mathcal{E}} = \tau^{(1)} - \psi^{(1)}_{\mathcal{O}} - \phi^{(1)}_{\mathcal{E}} - k^{(1)0}_{\mathcal{E}} \tag{5.2.14}$$

and at second order

$$\begin{aligned}\frac{\delta T^{(2)}}{T} =& \tau^{(2)} + \left[k^{(2)0} + \frac{1}{2}\phi^{(2)} - \frac{1}{2}\left(\phi^{(1)}\right)^2 + k^{(1)0}\phi^{(1)}\right]_{\mathcal{E}}^{\mathcal{O}} + \left.\frac{dk^{(1)0}}{d\lambda}\right|_{\mathcal{E}} x^{(1)0}_{\mathcal{E}} + \\ & - \left(\phi^{(1)} + k^{(1)0} - \tau^{(1)}\right)_{\mathcal{E}} \left(\phi^{(1)} + k^{(1)0}\right)_{\mathcal{E}}^{\mathcal{O}}\end{aligned} \tag{5.2.15}$$

where the term $\left.\frac{dk^{(1)0}}{d\lambda}\right|_{\mathcal{E}} x^{(1)0}_{\mathcal{E}}$ is due to the fact that we must expand all the quantities around the background geodesic, and $x^{(1)0}_{\mathcal{E}}$ is the difference in affine parameter between the point where zeroth and first order geodesics intersect the hypersurface of emission.

The next step is to find the solution of the geodesic equation in the non-linear case. As background geodesics we take the straight lines of eq. (5.1.11), and we place the boundary conditions at the observer's point

$$x^{\mu}(\lambda_{\mathcal{O}}) - x^{(0)\mu}(\lambda_{\mathcal{O}}) = 0\,, \quad k^{i}(\lambda_{\mathcal{O}}) - k^{(0)i}(\lambda_{\mathcal{O}}) = 0\,. \tag{5.2.16}$$

The null vector condition $g_{\mu\nu}k^{\mu}k^{\nu} = 0$ gives

$$\delta_{ij}k^{i}k^{j} = e^{2(\Phi+\Psi)}\left(k^{0}\right)^{2}\,, \tag{5.2.17}$$

from which we have, at the observer's point,

$$k^{0}(\lambda_{\mathcal{O}}) = e^{-(\Phi_{\mathcal{O}}+\Psi_{\mathcal{O}})}\,. \tag{5.2.18}$$

Finally, we can write down the geodesic equation for the metric (5.2.1)

$$\begin{aligned}\frac{dk^{0}}{d\lambda} &= -\,\Gamma^{0}_{\alpha\beta}k^{\alpha}k^{\beta} = -2\Gamma^{0}_{i0}k^{i}k^{0} - \Gamma^{0}_{00}k^{0}k^{0} - \Gamma^{0}_{ij}k^{i}k^{j} = \\ &= -\,2\left(\partial_{i}\Phi\right)k^{i}k^{0} - \Phi'\left(k^{0}\right)^{2} + \Psi' e^{-2(\Phi+\Psi)}\,\delta_{ij}k^{i}k^{j} = \\ &= -\,\left[2\left(\partial_{i}\Phi\right)k^{i} + \Phi' k^{0}\right]k^{0} + \Psi'\left(k^{0}\right)^{2} = \\ &= -\,\left[2\left(\frac{d\Phi}{d\lambda} - \Phi k'^{0}\right) + \Phi' k^{0}\right]k^{0} + \Psi'\left(k^{0}\right)^{2} = \\ &= -\,2\frac{d\Phi}{d\lambda}k^{0} + \left(\Phi' + \Psi'\right)\left(k^{0}\right)^{2}\end{aligned} \tag{5.2.19}$$

where we have used eq. (5.2.17) and the relation $\frac{d\Phi}{d\lambda} = k^{0}\,\partial_{0}\Phi + k^{i}\,\partial_{i}\Phi$.

The solution is

$$k^{0}(\lambda) = e^{-2\Phi(\lambda)}\left[\frac{e^{-2\Phi(\lambda_{\mathcal{O}})}}{k^{0}(\lambda_{\mathcal{O}})} - \int_{\lambda_{\mathcal{O}}}^{\lambda} e^{-2\Phi(\lambda')}\left(\Phi'(\lambda') + \Psi'(\lambda')\right)d\lambda'\right]^{-1} \tag{5.2.20}$$

where the potentials are to be computed along the true geodesic.

To make contact with the second order result, in addition to the expansion in perturbative orders we need to perform a sort of Taylor expansion around the background geodesic for the potentials, in order to express all the quantities on the background geodesic. For example, if we have the non-linear quantity $\alpha(x)$ computed on the true geodesic, its expansion up to second order will be $\alpha(x) = \alpha^{(1)}(x^{(0)}) + \partial_{\mu}\alpha^{(1)}(x^{(0)})\,x^{(1)\mu} + \alpha^{(2)}(x^{(0)})$.

Therefore we find

$$k^0(\lambda_{\mathcal{E}}) \simeq \left[1 + 2\left(\Phi_{\mathcal{O}} - \Phi_{\mathcal{E}}\right) + 2\left(\Phi_{\mathcal{O}} - \Phi_{\mathcal{E}}\right)^2\right]\left(1 + k^{(1)0}_{\mathcal{O}} + k^{(2)0}_{\mathcal{O}}\right) \times \left[1 + k^0_{\mathcal{O}} e^{2\Phi_{\mathcal{O}}} \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} e^{-2\Phi}\left(\Phi' + \Psi'\right) d\lambda + \left(k^0_{\mathcal{O}}\right)^2 e^{4\Phi_{\mathcal{O}}}\left(\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} e^{-2\Phi}\left(\Phi' + \Psi'\right) d\lambda\right)^2\right] = \\ = \left[1 + 2\,\phi^{(1)}\Big|_{\mathcal{E}}^{\mathcal{O}} + \phi^{(2)}\Big|_{\mathcal{E}}^{\mathcal{O}} + 4\left(\phi^{(1)}_{\mathcal{E}} - 2\phi^{(1)'}x^{(1)0}\right)^2 - 4\phi^{(1)}_{\mathcal{O}}\phi^{(1)}_{\mathcal{E}}\right] \times \left(1 + k^{(1)0}_{\mathcal{O}} + k^{(2)0}_{\mathcal{O}}\right)\Big\{1 + I_1 + \left(k^{(1)0}_{\mathcal{O}} + 2\phi^{(1)}_{\mathcal{O}}\right) I_1 + I_1^2 + \\ + \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}}\left[-2\phi^{(1)}A^{(1)'} + \frac{1}{2}A^{(2)'} + A^{(1)''}x^{(1)0} + A^{(1)'}_{,i}x^{(1)i} - 2\phi^{(1)'}\phi^{(1)} + 2\psi^{(1)'}\psi^{(1)}\right] d\lambda\Big\} \tag{5.2.21}$$

where $A^{(n)} \equiv \phi^{(n)} + \psi^{(n)}$ and $I_1 \equiv \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} A^{(1)'} d\lambda$.

The first order term is

$$k^{(1)0}_{\mathcal{E}} = 2\phi^{(1)}_{\mathcal{O}} - 2\phi^{(1)}_{\mathcal{E}} + k^{(1)0}_{\mathcal{O}} + I_{ISW} = \phi^{(1)}_{\mathcal{O}} - \psi^{(1)}_{\mathcal{O}} - 2\phi^{(1)}_{\mathcal{E}} + I_{ISW}\,, \tag{5.2.22}$$

and the second order one is

$$k^{(2)0}_{\mathcal{E}} = k^{(2)0}_{\mathcal{O}} + \phi^{(2)}\Big|_{\mathcal{E}}^{\mathcal{O}} + 4\left(\phi^{(1)}_{\mathcal{E}}\right)^2 - 2\phi^{(1)'}_{\mathcal{E}}x^{(1)0}_{\mathcal{E}} - 2\phi^{(1)}_{\mathcal{O}}\phi^{(1)}_{\mathcal{E}} + \\ + 2I_1\left(\phi^{(1)}_{\mathcal{O}} - \psi^{(1)}_{\mathcal{O}} - 2\phi^{(1)}_{\mathcal{E}}\right) - 2\left(\phi^{(1)}_{\mathcal{O}}\right)^2 - 2\psi^{(1)}_{\mathcal{O}}\phi^{(1)}_{\mathcal{O}} + 2\psi^{(1)}_{\mathcal{O}}\phi^{(1)}_{\mathcal{E}} + \\ + \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}}\left[\frac{1}{2}A^{(2)'} + A^{(1)''}x^{(1)0} + 2A^{(1)'}I_1 - 4\phi^{(1)'}\phi^{(1)} + 2\psi^{(1)'}\psi^{(1)} - 2\psi^{(1)'}\phi^{(1)}\right] d\lambda\,. \tag{5.2.23}$$

Let us see what is the expression for the intrinsic temperature anisotropy $T_{\mathcal{E}}$ in our formalism. For this purpose, we generalize the adiabaticity condition at any order in perturbation theory.

First, we define the non linear quantity

$$\mathcal{F} \equiv \ln(ae^{-\Psi}) + \frac{1}{3}\int^{\rho} \frac{d\tilde{\rho}}{\tilde{\rho} + \tilde{P}} \tag{5.2.24}$$

where ρ and P are the energy density and pressure in the non linear case. The quantity in eq. (5.2.24) is conserved in time: in fact, by using the (non linear) continuity equation on large scales

$$\rho' = -3\left(\mathcal{H} - \Psi'\right)(\rho + P) \tag{5.2.25}$$

we find

$$\mathcal{F}' = \mathcal{H} - \Psi' + \frac{1}{3}\frac{\rho'}{\rho + P} = 0\,. \tag{5.2.26}$$

The perturbation $\delta\mathcal{F}$ is a gauge-invariant quantity representing the non linear generalization of the curvature perturbation ζ. Indeed, by expanding it at first order we obtain $\zeta^{(1)} = -\psi^{(1)} - \frac{\delta^{(1)}\rho}{\rho'}$, and at second order we obtain the $\zeta^{(2)}$ defined in eq. (3.4.5).

At non-linear level, the first-order adiabaticity condition $\zeta_m^{(1)} = \zeta_\gamma^{(1)}$ generalizes to $\delta\mathcal{F}_m = \delta\mathcal{F}_\gamma$: explicitly we have

$$\frac{1}{3}\int\frac{d\rho_m}{\rho_m} = \frac{1}{4}\int\frac{d\rho_\gamma}{\rho_\gamma} \Longrightarrow \ln\rho_m = \ln\rho_\gamma^{\frac{3}{4}}\,. \tag{5.2.27}$$

To relate the photon energy density to the gravitational potentials, we use the $(0-0)$ component of Einstein equations at recombination, where we consider full matter domination:

$$e^{-2\Phi}\left(\mathcal{H} - \Psi'\right) = \frac{8\pi G}{3}\rho_m\,. \tag{5.2.28}$$

Neglecting the term Ψ' since the potentials remain constant on large scales in the matter dominated era, by comparison with the background $(0-0)$ Einstein equation we find

$$\rho_m = \rho_m^{(0)} e^{-2\Phi}\,. \tag{5.2.29}$$

There is indeed a small amount of radiation at the recombination epoch, which gives rise to the early integrated Sachs-Wolfe effect: so, strictly speaking, we are not allowed to neglect ρ_γ and Ψ' in our equation. However, we can consider some moment after recombination in the full matter dominated era, and then taking into account separately the evolution of potentials around the recombination epoch.

Finally, since $T_\mathcal{E} \propto \rho_\gamma^{\frac{1}{4}}$, from eqs. (5.2.27) and (5.2.29) we have

$$T_\mathcal{E} = T_\mathcal{E}^{(0)} e^{-\frac{2}{3}\Phi} \tag{5.2.30}$$

which is the non linear generalization of the intrinsic temperature anisotropies for adiabatic perturbations.

Neglecting integrated effects, the non-linear generalization of the Sachs-Wolfe effect is

$$\frac{\delta_{np}T}{T} = e^{\frac{1}{3}\Phi} - 1\,, \tag{5.2.31}$$

which reproduces the known perturbative results at first and second order.

5.3 Non-perturbative Einstein equations

Now we write down the Einstein equations for the metric (5.2.1), showing that they correctly give the perturbed equations up to second order.

The $(0-0)$ equation is

$$\frac{e^{-2\Phi}}{a^2}\left(\mathcal{H}-\Psi'\right)^2=\frac{8\pi G}{3}\sum_n\left[\frac{e^{-2\Phi}}{a^2}\rho_n(1+w_n)\,(u_0)^2-w_n\rho_n\right], \tag{5.3.1}$$

where the sum is over all the fluid components present at a given time, including dark energy with $\rho_\Lambda=\frac{\Lambda}{8\pi G}$, $w_\Lambda=-1$.

Using the normalization condition $g^{\mu\nu}u_\mu u_\nu=-1$ we find an expression for u_0:

$$u_0=-e^{\Phi}\left(a^2+e^{2\Psi}u^l u_l\right)^{\frac{1}{2}}\simeq -ae^{\Phi} \tag{5.3.2}$$

where we have neglected the term $u^l u_l$ In fact, since vector modes on large scales can be neglected, we consider only scalar velocities $u_l=\partial_l u$, so the term $u^l u_l$ contains two spatial gradients and it can be dropped.

Therefore eq. (5.3.1) can be written as

$$\frac{e^{-2\Phi}}{a^2}\left(\mathcal{H}-\Psi'\right)^2=\frac{8\pi G}{3}\sum_n\rho_n\,. \tag{5.3.3}$$

The $(0-i)$ equation is

$$\partial_i\Psi'+\left(\mathcal{H}-\Psi'\right)\partial_i\Phi=-4\pi Gae^{\Phi}\sum_n\rho_n(1+w_n)u_i \tag{5.3.4}$$

where we have made use of eq. (5.3.2). Hence we solve for the spatial velocities:

$$u_i=-\frac{1}{a\,4\pi G}\frac{e^{-\Phi}}{\sum_n\rho_n(1+w_n)}\left[\partial_i\Psi'+\left(\mathcal{H}-\Psi'\right)\partial_i\Phi\right]\,. \tag{5.3.5}$$

The $(i-j)$ traceless equation is given by

$$\begin{aligned}&\frac{e^{2\Psi}}{a^2}\Big[\partial^i\partial_j\Psi-\frac{1}{3}\nabla^2\Psi\delta^i_j-\partial^i\partial_j\Phi+\frac{1}{3}\nabla^2\Phi+\partial^i\Psi\partial_j\Psi-\frac{1}{3}\partial^l\Psi\partial_l\Psi\delta^i_j+\\&-\partial^i\Phi\partial_j\Phi+\frac{1}{3}\partial^l\Phi\partial_l\Phi\delta^i_j-\partial^i\Phi\partial_j\Psi-\partial^i\Psi\partial_j\Phi+\frac{2}{3}\partial^l\Phi\partial_l\Psi\delta^i_j\Big]=\\=&8\pi G\frac{e^{2\Psi}}{a^2}\sum_n\rho_n(1+w_n)\left(u^i u_k-\frac{1}{3}u^l u_l\delta^i_j\right)\end{aligned} \tag{5.3.6}$$

from which, substituting eq. (5.3.5) we obtain

$$\begin{aligned}
&\partial^i\partial_j\Psi - \frac{1}{3}\nabla^2\Psi\delta^i_j - \partial^i\partial_j\Phi + \frac{1}{3}\nabla^2\Phi + \partial^i\Psi\partial_j\Psi - \frac{1}{3}\partial^l\Psi\partial_l\Psi\delta^i_j + \\
&- \partial^i\Phi\partial_j\Phi + \frac{1}{3}\partial^l\Phi\partial_l\Phi\delta^i_j - \partial^i\Phi\partial_j\Psi - \partial^i\Psi\partial_j\Phi + \frac{2}{3}\partial^l\Phi\partial_l\Psi\delta^i_j = \\
=&\frac{1}{a^2 2\pi G}\frac{e^{-2\Phi}}{\sum_n \rho_n(1+w_n)}\left[\left(\mathcal{H}-\Psi'\right)^2\left(\partial^i\Phi\partial_k\Phi - \frac{1}{3}\partial^l\Phi\partial_l\Phi\delta^i{}_j\right) + \right.\\
&\left. + \left(\mathcal{H}-\Psi'\right)\left(\partial^i\Psi'\partial_k\Phi + \partial^i\Phi\partial_k\Psi' - \frac{2}{3}\partial^l\Psi'\partial_l\Phi\delta^i{}_j\right) + \partial^i\Psi'\partial_k\Psi' - \frac{1}{3}\partial^l\Psi'\partial_l\Psi'\delta^i{}_j\right].
\end{aligned} \tag{5.3.7}$$

The trace part of the $(i-j)$ equations gives

$$\begin{aligned}
&\frac{e^{-2\Phi}}{a^2}\left[\left(\mathcal{H}-\Psi'\right)\left(-\dot{H}+3\Psi'+2\Phi'\right) - 2\mathcal{H}' + 2\Psi''\right] + \\
&+\frac{e^{2\Psi}}{a^2}\left[2\nabla^2\left(\Phi-\Psi\right) + \partial^l\Psi\partial_l\Psi + 2\partial^l\Phi\partial_l\Phi - \partial^l\Phi\partial_l\Psi\right] = \\
=&\frac{e^{2\Psi}}{a^2}\sum_n\left[\rho_n(1+w_n)\left(u^l u_l\right) + 3w_n\rho_n\right] = \\
=&\frac{e^{2(\Psi-\Phi}}{2\pi G a^4}\frac{1}{\sum_n \rho_n(1+w_n)}\left[\partial_i\Psi' + \left(\mathcal{H}-\Psi'\right)\partial_i\Phi\right]^2 + 24\pi G\sum_n \rho_n w_n\,.
\end{aligned} \tag{5.3.8}$$

In principle, one needs to solve these equations to obtain an expression for the integrated ISW effects, taking into account all the contributions to the observed non-Gaussianity.

Unfortunately, they are too complicated to be solved explicitly: so, in the general case, one must expand the potentials to the desired order and proceed with a perturbative calculation, as usual.

Indeed, it is worth noting that it is easier to recover the perturbed equations starting from these and simply expanding Φ and Ψ, instead of computing the Einstein tensor for the perturbed metric. For example, it would be relatively straightforward going to third order, in order to provide an expression for the non-linearity parameter g^{ϕ}_{NL} which enters in the computation of the trispectrum on large scales.

5.3.1 Bispectrum from the Sachs-Wolfe effect

However, there is a special case for which we can find a solution for the potentials in terms on the initial conditions provided by inflation: in fact, for a matter dominated universe, the potentials stay constant on large scales, so we can neglect the time derivatives and the integrated effects. Using a path integral technique, we will provide an expression for the bi- and three-spectrum on large scales, taking into account only the Sachs-Wolfe effect [38].

In this case eq. (5.3.3) reads

$$\rho_m = \frac{3e^{-2\Phi}}{a^2 8\pi G}\mathcal{H}^2 \,. \tag{5.3.9}$$

Substituting this expression into eq. (5.3.7) and dropping time derivatives, we obtain

$$\begin{aligned}
&\partial^i\partial_j\Psi - \frac{1}{3}\nabla^2\Psi\delta^i_j - \partial^i\partial_j\Phi + \frac{1}{3}\nabla^2\Phi + \partial^i\Psi\partial_j\Psi - \frac{1}{3}\partial^l\Psi\partial_l\Psi\delta^i_j + \\
&-\partial^i\Phi\partial_j\Phi + \frac{1}{3}\partial^l\Phi\partial_l\Phi\delta^i_j - \partial^i\Phi\partial_j\Psi - \partial^i\Psi\partial_j\Phi + \frac{2}{3}\partial^l\Phi\partial_l\Psi\delta^i_j = \\
=&\frac{4}{3}\left(\partial^i\Phi\partial_k\Phi - \frac{1}{3}\partial^l\Phi\partial_l\Phi{\delta^i}_j\right) \,,
\end{aligned} \tag{5.3.10}$$

and by applying the operator $\partial_i\partial^j$ we finally have

$$\begin{aligned}
\nabla^4\left(\Psi-\Phi\right) =& \frac{7}{2}\partial_i\partial^j\left(\Phi^{,i}\Phi_{,j}\right) + 3\partial_i\partial^j\left(\Phi^{,i}\Psi_{,j}\right) - \nabla^2\left(\Phi^{,l}\Psi_{,l}\right) + \\
&-\frac{7}{6}\nabla^2\left(\Phi^{,l}\Phi_{,l}\right) - \frac{3}{2}\partial_i\partial^j\left(\Psi^{,i}\Psi_{,j}\right) + \frac{1}{2}\nabla^2\left(\Psi^{,l}\Psi_{,l}\right)
\end{aligned} \tag{5.3.11}$$

which is the non-linear generalization of the constraint between ϕ and ψ in the Poisson gauge.

It is convenient now to write this equation as $\Psi = \Phi + \mathcal{K}[\Phi,\Psi]$ where $\mathcal{K}$ is the kernel obtained by acting with the operator ∇^{-4} on the RHS of the previous equation.

The non-linear curvature perturbation reads

$$\zeta \equiv \delta\mathcal{F} = -\Psi + \frac{1}{3}\ln\frac{\rho_m}{\rho_m^{(0)}} = -\Psi - \frac{2}{3}\Phi = -\frac{5}{3}\Phi - \mathcal{K}[\Phi,\Psi] \tag{5.3.12}$$

where we have used eq.(5.2.29).

So we can write the temperature anisotropy (5.2.31) as

$$\frac{\delta_{np}T}{T} = e^{-\frac{1}{5}\zeta - \frac{1}{5}\mathcal{K}} - 1 \tag{5.3.13}$$

which is our starting point for the evaluation of the n-point correlation function for the large-scale CMB anisotropies, using a functional integral tecnique.

Let us see how to evaluate the n-point correlation function of $e^{\varphi(\mathbf{x})}$ where $\varphi(\mathbf{x})$ is a Gaussian random field. We have:

$$\begin{aligned}
\left\langle e^{\varphi(\mathbf{x}_1)}\dots e^{\varphi(\mathbf{x}_n)}\right\rangle =& \int\mathcal{D}\varphi\mathcal{P}[\varphi]e^{\int J(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}} = \\
=&e^{\frac{1}{2}\int d\mathbf{x}d\mathbf{y}J(\mathbf{x})\langle\varphi(\mathbf{x})\varphi(\mathbf{y})\rangle J(\mathbf{y})} = e^{\frac{1}{2}\sum_{i,j}\langle\varphi(\mathbf{x}_i)\varphi(\mathbf{x}_j)\rangle}
\end{aligned} \tag{5.3.14}$$

where $J(\mathbf{x}) = \sum_{i=1}^{n} \delta(\mathbf{x} - \mathbf{x}_i)$, $\mathcal{D}\varphi$ is the functional measure and $\mathcal{P}[\varphi]$ is the Gaussian probability density functional.

The correlation function of $e^{\varphi(\mathbf{x})}\varphi(\mathbf{y})$ will be

$$\left\langle e^{\varphi(\mathbf{x}_1)}\varphi(\mathbf{y})\right\rangle = \int \mathcal{D}\varphi\mathcal{P}[\varphi]\varphi(\mathbf{y})e^{\int J(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}} = \frac{\delta}{\delta J(\mathbf{y})}\int \mathcal{D}\varphi\mathcal{P}[\varphi]e^{\int J(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}} = \\ = \frac{\delta}{\delta J(\mathbf{y})} e^{\frac{1}{2}\int d\mathbf{x}d\mathbf{y}J(\mathbf{x})\langle\varphi(\mathbf{x})\varphi(\mathbf{y})\rangle J(\mathbf{y})} = e^{\frac{1}{2}\langle\varphi(\mathbf{x})^2\rangle} 2\left\langle \varphi(\mathbf{x})\varphi(\mathbf{y})\right\rangle . \tag{5.3.15}$$

Now, the gravitational potential Φ in (5.2.31) is not a Gaussian variable: the curvature perturbation in general will be non-Gaussian, too, but we will split it into a linear Gaussian part plus non linear (non-Gaussian) corrections, that will be put into the kernel $\mathcal{K}$.

So, our task is to compute the correlation functions of the temperature anisotropy (5.3.13). Applying the techniques of quantum field theory, the kernel $\mathcal{K}$ can be treated as an interaction term, and we apply an iterative procedure to express it in powers of the Gaussian field ζ, using the definition of $\mathcal{K}[\Phi,\Psi]$ and eq. (5.3.12).

Now, we start from equation (5.3.12) written in the form

$$\Phi = -\frac{3}{5}\zeta - \frac{3}{5}\mathcal{K}[\Phi, -\zeta - \frac{2}{3}\Phi] \tag{5.3.16}$$

and we solve it perturbatively to find $\Phi[\zeta]$.

The zeroth and first-order terms are given by

$$\Phi^{(0)} = -\frac{3}{5}\zeta\,, \quad \Phi^{(1)} = -\frac{3}{5}\zeta - \frac{3}{5}\mathcal{K}[\Phi^{(0)}, -\zeta - \frac{2}{3}\Phi^{(0)}] = -\frac{3}{5}\zeta - \left(\frac{3}{5}\right)^3 \mathcal{K}[\zeta^2]\,. \tag{5.3.17}$$

In general we have the iterative solution

$$\begin{aligned} \Phi^{(2n)} =& \Phi^{(2n-1)} + \mathcal{K}_1[\Phi^{(0)}, \Phi^{(2n-2)} - \Phi^{(2n-1)}] + \\ & + \sum_{m=0}^{n-2} \mathcal{K}_1[\Phi^{(m)} - \Phi^{(m+1)}, \Phi^{(2n-m-3} - \Phi^{(2n-m-2}]\,, \end{aligned} \tag{5.3.18}$$

$$\begin{aligned} \Phi^{(2n+1)} =& \Phi^{(2n)} + \mathcal{K}_1[\Phi^{(0)}, \Phi^{(2n-1)} - \Phi^{(2n)}] + \\ & + \sum_{m=0}^{n-2} \mathcal{K}_1[\Phi^{(m)} - \Phi^{(m+1)}, \Phi^{(2n-m-2} - \Phi^{(2n-m-1}] + \mathcal{K}_2[\left(\Phi^{(n-1)} - \Phi^{(n)}\right)^2]\,, \end{aligned} \tag{5.3.19}$$

where we have introduced the bilinear operators

$$\mathcal{K}_1[f,g] \equiv \nabla^{-4}\left[6\partial_i\partial^j(f^{,i}g_{,j}) - 2\nabla^2(f^{,l}g_{,l})\right]\,, \tag{5.3.20}$$

$$\mathcal{K}_2[f,g] \equiv -\nabla^{-4}\left[\frac{1}{2}\partial_i\partial^j(f^{,i}g_{,j}) - \frac{1}{2}\nabla^2(f^{,l}g_{,l})\right]\,, \tag{5.3.21}$$

in terms of which we have $\mathcal{K}[\Phi,\Psi] = \frac{7}{12}\mathcal{K}_1[\Phi,\Phi] - \frac{1}{4}\mathcal{K}_1[\Psi,\Psi] + \frac{1}{2}\mathcal{K}_1[\Phi,\Psi]$.

To compute the 3-point function, we need to expand the kernel up to second order in ζ, and this means to know $\Phi^{(2)}$:

$$\Phi^{(2)} = -\frac{3}{5}\zeta - \left(\frac{3}{5}\right)^3 \mathcal{K}[\zeta^2] + \mathcal{K}_1\left[-\frac{3}{5}\zeta, \left(\frac{3}{5}\right)^3 \mathcal{K}[\zeta^2]\right]. \tag{5.3.22}$$

In order to compute the n-point correlation functions, we define the generating functional

$$Z[J] = \int \mathcal{D}[\zeta]\mathcal{P}[\zeta] e^{i\int d\mathbf{x} J(\mathbf{x})\left(e^{-\zeta/5-\mathcal{K}[\zeta]/5}-1\right)} \tag{5.3.23}$$

with $J(\mathbf{x})$ an external source, $\mathcal{P}[\zeta]$ is the Gaussian PDF

$$\mathcal{P}[\zeta] = \frac{e^{-\frac{1}{2}\int d\mathbf{x}d\mathbf{y}\zeta(\mathbf{x})G(\mathbf{x},\mathbf{y})\zeta(\mathbf{y})}}{\int \mathcal{D}[\zeta] e^{-\frac{1}{2}\int d\mathbf{x}d\mathbf{y}\zeta(\mathbf{x})G(\mathbf{x},\mathbf{y})\zeta(\mathbf{y})}} \tag{5.3.24}$$

where $G(\mathbf{x},\mathbf{y})$ is the functional inverse of $\langle\zeta(\mathbf{x})\zeta(\mathbf{y})\rangle$ and $\mathcal{D}[\zeta]$ is a functional measure such that $\int \mathcal{D}[\zeta]\mathcal{P}[\zeta] = 1$.

The correlation functions will be obtained by functional derivation with respect to J:

$$\left\langle \left(e^{-\zeta_1/5-\mathcal{K}[\zeta_1]/5} - 1\right) \dots \left(e^{-\zeta_n/5-\mathcal{K}[\zeta_n]/5} - 1\right) \right\rangle = i^{-n} \left.\frac{\delta^n Z[J]}{\delta J(\mathbf{x}_1)\dots\delta J(\mathbf{x}_n)}\right|_{J=0} \equiv$$
$$\equiv Z^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n). \tag{5.3.25}$$

The connected correlation functions are obtained by defining the new functional $W[J] \equiv \ln Z[J]$, with the analog formula

$$\left\langle \left(e^{-\zeta_1/5-\mathcal{K}[\zeta_1]/5} - 1\right) \dots \left(e^{-\zeta_n/5-\mathcal{K}[\zeta_n]/5} - 1\right) \right\rangle_{\text{conn.}} = i^{-n} \left.\frac{\delta^n W[J]}{\delta J(\mathbf{x}_1)\dots\delta J(\mathbf{x}_n)}\right|_{J=0} \equiv$$
$$\equiv W^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n). \tag{5.3.26}$$

The computation of the generating functional is made, as in quantum field theory, employing a perturbative expansion around some known solution, which in our case corresponds to the connected correlation functions when the kernel $\mathcal{K}$ vanishes.

Let us show the explicit computation for the bispectrum. To compute the functional derivatives with respect to J we find it convenient to introduce another external source $\lambda(\mathbf{x})$, defining the new generating functional

$$Z[J,\lambda] = \int \mathcal{D}[\zeta]\mathcal{P}[\zeta] e^{i\int d\mathbf{x} J(\mathbf{x})\left(e^{-\zeta/5-\mathcal{K}[\zeta]/5}-1\right)} e^{i\int d\mathbf{x}\lambda(\mathbf{x})\zeta(\mathbf{x})}. \tag{5.3.27}$$

Functional of this form are common in quantum field theory when treating with composite operators.

Clearly, the correlation functions generated by $Z[J,\lambda]$ are given by

$$\left\langle \left(e^{-\zeta(\mathbf{x}_1)/5-\mathcal{K}[\zeta(\mathbf{x}_1)]/5}-1\right)\dots\left(e^{-\zeta(\mathbf{x}_n)/5-\mathcal{K}[\zeta(\mathbf{x}_n)]/5}-1\right)\zeta(\mathbf{y}_1)\dots\zeta(\mathbf{y}_m)\right\rangle = \\ = i^{-(n+m)}\left.\frac{\delta^{n+m}Z[J,\lambda]}{\delta J(\mathbf{x}_1)\dots\delta J(\mathbf{x}_n)\delta\lambda(\mathbf{y}_1)\dots\delta\lambda(\mathbf{y}_m)}\right|_{J,\lambda=0}. \quad (5.3.28)$$

If we write

$$e^{\varphi+\mathcal{K}[\varphi]}-1=\sum_{n=1}^{\infty}\frac{\mathcal{K}^n[\varphi]}{n!}e^{\varphi}+(e^{\varphi}-1)\,, \quad (5.3.29)$$

the generating functional can be put in the form

$$Z[J,\lambda]=\exp i\int d\mathbf{x}J(\mathbf{x})\sum_n\frac{\mathcal{K}^n[\frac{1}{i}\frac{\delta}{\delta\lambda}]}{n!}e^{\frac{1}{i}\frac{\delta}{\delta\lambda}}Z_0[J,\lambda] \quad (5.3.30)$$

where $Z_0[J,\lambda]$ is the generating functional when $\mathcal{K}[\varphi]=0$:

$$Z_0[J,\lambda]=\int\mathcal{D}[\varphi]\mathcal{P}[\varphi]e^{i\int d\mathbf{x}J(\mathbf{x})(e^{\varphi(\mathbf{x})}-1)}e^{i\int d\mathbf{x}\lambda(\mathbf{x})\varphi(\mathbf{x})}\equiv e^{W_0[J,\lambda]}. \quad (5.3.31)$$

So we have isolated the "interaction" term from the "free" term. Now we can write

$$W[J,\lambda]=W_0[J,\lambda]+\ln\left\{1+e^{-W_0}\left[\exp i\int d\mathbf{x}J(\mathbf{x})\sum_n\frac{\mathcal{K}^n[\frac{1}{i}\frac{\delta}{\delta\lambda}]}{n!}e^{\frac{1}{i}\frac{\delta}{\delta\lambda}}-1\right]e^{W_0}\right\} \quad (5.3.32)$$

which, by functional derivation with respect to J, generates the connected correlation functions for the temperature anisotropies.

At this point we perform a perturbative expansion in the small r. m. s amplitude of the perturbations, $\langle\varphi^2\rangle^{\frac{1}{2}}\ll 1$, and use the fact that the kernal is given by the iterative procedure described above. We write $\mathcal{K}[\varphi]=a\star\varphi^2+b\star\varphi^3$, where the $\star$ denotes a convolution in configuration space: for the moment, however, we will treat a and b as constant coefficient.

For the bispectrum we just need the quadratic piece $a\varphi^2$. So, taking the logarithm term in eq. (5.3.32), we have

$$\ln(1+\Delta)\simeq\Delta\simeq iae^{-W_0}\int d\mathbf{x}J(\mathbf{x})\left(\frac{1}{i}\frac{\delta}{\delta\lambda}\right)^2e^{W_0}= \\ =\frac{1}{i}a\int d\mathbf{x}J(\mathbf{x})\left[\frac{\delta^2W_0}{\delta\lambda^2(\mathbf{x})}+\left(\frac{\delta W_0}{\delta\lambda(\mathbf{x})}\right)^2\right] \quad (5.3.33)$$

where Δ is read from eq. (5.3.32), and we have put $e^{\frac{1}{i}\frac{\delta}{\delta\lambda}}=1+\frac{\delta}{\delta\lambda}+\dots\simeq 1$.

Therefore the bispectrum is given by:

$$\begin{aligned} W^{(3)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) =& i^{-3}\frac{\delta^3 W[J,\lambda]}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)\delta J(\mathbf{x}_3)} = \\ =& \left\langle \left(e^{\varphi(\mathbf{x}_1)}-1\right)\left(e^{\varphi(\mathbf{x}_3)}-1\right)\left(e^{\varphi(\mathbf{x}_3)}-1\right)\right\rangle_{\text{conn.}} + \\ & + i^{-3}\left.\frac{\delta^3 \Delta[J,\lambda]}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)\delta J(\mathbf{x}_3)}\right|_{J,\lambda=0} \end{aligned} \tag{5.3.34}$$

where we have to compute the last contribution, and to this end we need an expression for $W_0[J,\lambda]$. Now, since the functional derivatives of W_0 with respect to J and λ give respectively the correlation functions of $(e^{\varphi}1)$ and of φ, which are known, we can write

$$W_0[J,\lambda] = \sum_{n=1}^{\infty}\frac{i^n}{n!}\int d\mathbf{x}_1\ldots d\mathbf{x}_n \tilde{w}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\tilde{J}(\mathbf{x}_1)\ldots\tilde{J}(\mathbf{x}_n) \tag{5.3.35}$$

where $\tilde{J}(\mathbf{x}_i)$ can be either $J(\mathbf{x}_i)$ or $\lambda(\mathbf{x}_i)$, and $\tilde{w}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ are the corresponding connected correlation functions.

For the first term in Δ we find explicitly

$$\begin{aligned} &\frac{\delta^3}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)\delta J(\mathbf{x}_3)}\int d\mathbf{x} J(\mathbf{x})\left.\frac{\delta^2 W_0}{\delta\lambda^2(\mathbf{x})}\right|_{(J,\lambda)=0} = \\ &= \left.\frac{\delta^4 W_0}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)\delta^2\lambda(\mathbf{y}}\right|_{J,\lambda=0}\delta(\mathbf{y}-\mathbf{x}_3) + \text{cycl.} = \tilde{w}_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_3) + \text{cycl.} \end{aligned} \tag{5.3.36}$$

and for the second term

$$\begin{aligned} &\frac{\delta^3}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)\delta J(\mathbf{x}_3)}\int d\mathbf{x} J(\mathbf{x})\left.\left(\frac{\delta W_0}{\delta\lambda(\mathbf{x})}\right)^2\right|_{(J,\lambda)=0} = \\ &2\left.\frac{\delta^2 W_0}{\delta J(\mathbf{x}_1)\delta\lambda(\mathbf{y})}\right|_{(J,\lambda)=0}\left.\frac{\delta^2 W_0}{\delta J(\mathbf{x}_2)\delta\lambda(\mathbf{y})}\right|_{(J,\lambda)=0}\delta(\mathbf{y}-\mathbf{x}_3) + \text{cycl.} = \\ &= 2\tilde{w}_2(\mathbf{x}_1,\mathbf{x}_3)\tilde{w}_2(\mathbf{x}_2,\mathbf{x}_3) + \text{cycl.}\,. \end{aligned} \tag{5.3.37}$$

So the functional derivative of Δ reads

$$\begin{aligned} &i^{-3}\left.\frac{\delta^3\Delta[J,\lambda]}{\delta J(\mathbf{x}_1)\delta J(\mathbf{x}_2)\delta J(\mathbf{x}_3)}\right|_{J,\lambda=0} = \\ &\qquad a\sum_p\left[\tilde{w}_2(\mathbf{x}_{p1},\mathbf{x}_{p3})\tilde{w}_2(\mathbf{x}_{p2},\mathbf{x}_{p3}) + \frac{1}{2}\tilde{w}_4(\mathbf{x}_{p1},\mathbf{x}_{p2},\mathbf{x}_{p3},\mathbf{x}_{p3})\right] \end{aligned} \tag{5.3.38}$$

where the sum is over all the permutations (p_1, p_2, p_3) of indices $(1,2,3)$ (not only the cyclic ones), and we have used the following notations:

$$\tilde{w}_2(\mathbf{x}, \mathbf{y}) \equiv \left\langle (e^{\varphi(\mathbf{x})} - 1)\varphi(\mathbf{y}) \right\rangle_{\text{conn.}} \tag{5.3.39}$$

$$\tilde{w}_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}, \mathbf{x}) \equiv \left\langle (e^{\varphi(\mathbf{x}_1)} - 1)(e^{\varphi(\mathbf{x}_2)} - 1)\varphi(\mathbf{x})\varphi(\mathbf{x}) \right\rangle_{\text{conn.}} . \tag{5.3.40}$$

In the general case in which the kernel is a convolution in configuration space, that is

$$\mathcal{K}[\varphi] = \int d\mathbf{y}_1 d\mathbf{y}_2 K(\mathbf{y} - \mathbf{y}_1, \mathbf{y} - \mathbf{y}_2)\varphi(\mathbf{y})\varphi(\mathbf{y}) . \tag{5.3.41}$$

it is not difficult to see that the bispectrum is given by

$$\begin{aligned} W^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= \left\langle \left(e^{\varphi(\mathbf{x}_1)} - 1\right)\left(e^{\varphi(\mathbf{x}_3)} - 1\right)\left(e^{\varphi(\mathbf{x}_3)} - 1\right) \right\rangle_{\text{conn.}} + \\ &+ \sum_p \int d\mathbf{y}_1 d\mathbf{y}_2 K(\mathbf{x}_{p3} - \mathbf{y}_1, \mathbf{x}_{p3} - \mathbf{y}_2)\tilde{w}_2(\mathbf{x}_{p1}, \mathbf{y}_1)\tilde{w}_2(\mathbf{x}_{p2}, \mathbf{y}_2) + \\ &+ \frac{1}{2}\sum_p \int d\mathbf{y}_1 d\mathbf{y}_2 K(\mathbf{x}_{p3} - \mathbf{y}_1, \mathbf{x}_{p3} - \mathbf{y}_2)\tilde{w}_4(\mathbf{x}_{p1}, \mathbf{x}_{p2}, \mathbf{y}_1, \mathbf{y}_2) . \end{aligned} \tag{5.3.42}$$

In our case the kernel at second order can be written as

$$\mathcal{K}[\zeta] = \int d\mathbf{y}_1 d\mathbf{y}_2 K_2(\mathbf{x} - \mathbf{y}_1, \mathbf{x} - \mathbf{y}_2)\zeta(\mathbf{y}_1)\zeta(\mathbf{y}_2) , \tag{5.3.43}$$

where K_2 is the double inverse Fourier transform of the expression

$$\tilde{K}_2(\mathbf{k}_1, \mathbf{k}_2) = (a_{NL} - 1) + \frac{9}{5}\left[\frac{(\mathbf{k}_1 \cdot \mathbf{k}_3)(\mathbf{k}_2 \cdot \mathbf{k}_3)}{k^4} - \frac{1}{3}\frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)}{k^2}\right] \tag{5.3.44}$$

where $\mathbf{k}_3 = -(\mathbf{k}_1 + \mathbf{k}_2)$ and $k = |\mathbf{k}_3|$. We have added the term $(a_{NL} - 1)$ whose role is to parametrize the level of primordial non-Gaussianity in the variable ζ, which is written as $\zeta = \zeta_L + (a_{NL} - 1) \star \zeta_L^2$ with ζ_L a Gaussian random variable. Indeed, the ζ that appears in our formule is ζ_L.

For the bispectrum we then find

$$\begin{aligned} W^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= \left\langle \left(e^{-\zeta(\mathbf{x}_1)/5} - 1\right)\left(e^{-\zeta(\mathbf{x}_2)/5} - 1\right)\left(e^{-\zeta(\mathbf{x}_3)/5} - 1\right) \right\rangle_{\text{conn.}} + \\ &- 5\sum_p \int d\mathbf{y}_1 d\mathbf{y}_2 K_2(\mathbf{x}_{p3} - \mathbf{y}_1, \mathbf{x}_{p3} - \mathbf{y}_2) \times \\ &\times \left[\tilde{w}_2(\mathbf{x}_{p1}, \mathbf{y}_1)\tilde{w}_2(\mathbf{x}_{p2}, \mathbf{y}_2) + \frac{1}{2}\tilde{w}_4(\mathbf{x}_{p1}, \mathbf{x}_{p2}, \mathbf{y}_1, \mathbf{y}_2)\right] \end{aligned} \tag{5.3.45}$$

with

$$\tilde{w}_2(\mathbf{x}_1, \mathbf{x}_2) \equiv -\frac{1}{5}\left\langle \left(e^{\zeta_1/5} - 1\right)\zeta_2\right\rangle_{\text{conn.}} = -\frac{1}{25}e^{\frac{\langle\zeta^2\rangle}{50}}\langle\zeta_1\zeta_2\rangle \tag{5.3.46}$$

$$\tilde{w}_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \equiv \frac{1}{25}\left\langle \left(e^{\zeta_1/5} - 1\right)\left(e^{\zeta_2/5} - 1\right)\zeta_3\zeta_4\right\rangle_{\text{conn.}} =$$
$$= \frac{\frac{\langle\zeta^2\rangle}{25}}{3\times 25^2}\left(e^{\langle\zeta_1\zeta_2\rangle/25} - 1\right)\left(\langle\zeta_1\zeta_4\rangle + \langle\zeta_2\zeta_4\rangle\right)\left(\langle\zeta_1\zeta_3\rangle + \langle\zeta_2\zeta_3\rangle\right) + \text{cyclic}\,. \tag{5.3.47}$$

Now, if we expand the exponentials in eq. (5.3.45) up to second order, we immediately recover the expression obtained at second-order in perturbation theory for the non-linearity parameter f_{NL}. Remembering the usual Sachs-Wolfe formula $\frac{\delta T}{T} = -\frac{1}{3}\Phi = -\frac{1}{3}\left(\Phi_L + f_{NL}\star\Phi_L^2\right)$, and that $e^{-\zeta/5} = e^{-\Phi_L/3} \simeq 1 - \frac{1}{3}\Phi_L + \frac{1}{18}\Phi_L^2$, we obtain

$$f_{NL} = -\left[\frac{5}{3}(1 - a_{NL}) + \frac{1}{6}\right] + \left[3\frac{(\mathbf{k}_1\cdot\mathbf{k}_3)(\mathbf{k}_2\cdot\mathbf{k}_3)}{k^4} - \frac{(\mathbf{k}_1\cdot\mathbf{k}_2)}{k^2}\right], \tag{5.3.48}$$

With the procedure just described, we can obtain also the 4-point connected correlation function, which vanish in the case of Gaussian distributed perturbations and can help to constrain non-Gaussianity.

The kernel must be expanded up to third order, and can be written as the convolution

$$\mathcal{K}[\zeta] = \int d\mathbf{y}_1 d\mathbf{y}_2 K_2(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2)\zeta(\mathbf{y}_1)\zeta(\mathbf{y}_2) +$$
$$\int d\mathbf{y}_1 d\mathbf{y}_2 d\mathbf{y}_3 K_3(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{y}_3)\zeta(\mathbf{y}_1)\zeta(\mathbf{y}_2)\zeta(\mathbf{y}_3)\,, \tag{5.3.49}$$

where K_2 is defined by eq. (5.3.44), and K_3 is the triple inverse Fourier transform of the expression

$$\tilde{K}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = (b_{NL} - 1) + (a_{NL} - 1)\mathcal{A}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + \mathcal{C}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \tag{5.3.50}$$

with

$$\mathcal{A}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{6}{5}\left[\frac{(\mathbf{k}_1\cdot\mathbf{k}_4)}{k^4}(\mathbf{k}_3\cdot\mathbf{k}_4 + \mathbf{k}_2\cdot\mathbf{k}_4) + \right.$$
$$\left. + \frac{(\mathbf{k}_2\cdot\mathbf{k}_4)(\mathbf{k}_3\cdot\mathbf{k}_4)}{k^4} - \frac{1}{3}\frac{\mathbf{k}_1\cdot(\mathbf{k}_2 + \mathbf{k}_3) + \mathbf{k}_2\cdot\mathbf{k}_3}{k^2}\right] \tag{5.3.51}$$

$$\mathcal{C}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{54}{25}\frac{(\mathbf{k}_3\cdot\mathbf{k}_4)[(\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{k}_4]}{k^4}\times$$
$$\times\left\{\frac{[\mathbf{k}_1\cdot(\mathbf{k}_1 + \mathbf{k}_2)][\mathbf{k}_2\cdot(\mathbf{k}_1 + \mathbf{k}_2)]}{|\mathbf{k}_1 + \mathbf{k}_2|^4} - \frac{1}{3}\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{|\mathbf{k}_1 + \mathbf{k}_2|^2}\right\} + \text{cycl.} \tag{5.3.52}$$

where $\mathbf{k}_4 = -(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$ and $k = |\mathbf{k}_4|$.

Now, we can expand the exponential in the expression (5.3.13) up to third order, and using the kernel defined in eq. (5.3.49), we are able to provide an expression for the non-linearity parameter g_{NL} which enters into the trispectrum of the CMB anisotropies and is the coefficient of an expansion of the gravitational potential Φ up to third order:

$$\Phi = \Phi_L + f_{NL} \star (\Phi_L)^2 + g_{NL} \star (\Phi_L)^3 \,. \qquad (5.3.53)$$

We find the following expression:

$$g_{NL}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{25}{9}(b_{NL} - 1) + \frac{25}{9}(a_{NL} - 1)\mathcal{A}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + \frac{25}{9}\mathcal{C}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + \\ - \frac{5}{9}(a_{NL} - 1) + \frac{1}{54} - \frac{1}{3}\left\{ \frac{[\mathbf{k}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2)][\mathbf{k}_2 \cdot (\mathbf{k}_1 + \mathbf{k}_2)]}{|\mathbf{k}_1 + \mathbf{k}_2|^4} - \frac{1}{3}\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{|\mathbf{k}_1 + \mathbf{k}_2|^2} + \text{cycl.} \right\} . \qquad (5.3.54)$$

5.4 Second order radiation transfer function

We now turn to the calculation of the radiation transfer function on large scales, including all the relevant effects [39]. Using the computed multipoles $a_{lm} = a^L_{lm} + a^{NL}_{lm}$, splitted in a linear and a non-linear part, we will have an expression for the angular bispectrum as $\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle = \langle a^L_{l_1 m_1} a^L_{l_2 m_2} a^{NL}_{l_3 m_3} \rangle$ + perm..

We will study separately the late ISW, the early ISW and the second order tensor contribution, and we recall how to compute the second-order transfer function for the Sachs-Wolfe effect.

5.4.1 Late ISW effect

Evolution of the gravitational potentials

The background equations are the Friedmann equations

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 (\rho^{(0)}_m + \rho_\Lambda) = \frac{8\pi G}{3} a^2 \rho^{(0)}_m + a^2 \frac{\Lambda}{3} \qquad (5.4.1)$$

$$\mathcal{H}^2 + 2\mathcal{H}' = a^2 \Lambda \qquad (5.4.2)$$

which follow from the $(0-0)$ and the $(i-i)$ Einstein equation respectively.

At linear order, the traceless part of the $(i-j)$ equation gives $\phi^{(1)} = \psi^{(1)} = \varphi$, and its trace gives the evolution equation

$$\varphi'' + 3\mathcal{H}\varphi' + (2\mathcal{H}' + \mathcal{H}^2)\varphi = 0 \Longrightarrow \varphi'' + 3\mathcal{H}\varphi' + a^2\Lambda\varphi = 0 \,. \qquad (5.4.3)$$

We can write the growing mode solution to eq. (5.4.3) as

$$\varphi(\mathbf{x}, \eta) = g(\eta)\varphi_0(\mathbf{x}) \qquad (5.4.4)$$

where $\varphi_0(\mathbf{x})$ is the peculiar gravitational potential linearly extrapolated to the present time, and $g(\eta)$ is the so-called growth suppression factor, which can be found in refs(...), and that in the $\Lambda = 0$ case is $g(\eta) = 1$.

To find the evolution equation at second order, we start from eq. (5.3.8) and expand it at second order:

$$\begin{aligned}
&\psi^{(2)\prime\prime} + 3\mathcal{H}\psi^{(2)\prime} + a^2\Lambda\psi^{(2)} = \mathcal{H}\left(\psi^{(2)\prime} - \phi^{(2)\prime}\right) + \\
&+ a^2\Lambda\left(\psi^{(2)} - \phi^{(2)}\right) + \frac{1}{3}\nabla^2\left(\psi^{(2)} - \phi^{(2)}\right) + \\
&+ 4a^2\Lambda\varphi^2 + 8\mathcal{H}\varphi\varphi' + (\varphi')^2 + \frac{7}{3}(\partial^i\varphi\partial_i\varphi) + \frac{8}{3}\varphi\nabla^2\varphi + \\
&+ \frac{1}{6\pi G a^2\rho_m^{(0)}}\left[\partial^i\varphi'\partial_i\varphi' + \mathcal{H}(\partial^i\varphi\partial_i\varphi)' + \mathcal{H}^2(\partial^i\varphi\partial_i\varphi)\right] ,
\end{aligned} \tag{5.4.5}$$

where we have used the background $(0-0)$ Einstein equation $\mathcal{H}' + 2\mathcal{H}^2 = a^2\Lambda$.

To solve for the combination $\psi^{(2)} - \phi^{(2)}$, we make use of eq. (5.3.7) expanded at second order:

$$\nabla^{-4}\left(\psi^{(2)} - \phi^{(2)}\right) = -4\nabla^4(\varphi^2) + 3\partial_i\partial^j P^i_j - \nabla^2 P^i_i \tag{5.4.6}$$

where we have applied the operator $\partial_i\partial^j$ and we have defined

$$\begin{aligned}
P^i_j =& 2\partial^i\varphi\partial_j\varphi + \frac{1}{2\pi G a^2\rho_m^{(0)}}\left[\partial^i\varphi'\partial_j\varphi' + \mathcal{H}(\partial^i\varphi\partial_j\varphi)' + \mathcal{H}^2(\partial^i\varphi\partial_j\varphi)\right] = \\
=& \frac{2}{4\pi G a^2\rho_m^{(0)}}\left[\partial^i\varphi'\partial_j\varphi' + \mathcal{H}(\partial^i\varphi\partial_j\varphi)' + \frac{1}{2}\left(5\mathcal{H}^2 - a^2\Lambda\right)(\partial^i\varphi\partial_j\varphi)\right] .
\end{aligned} \tag{5.4.7}$$

We now substitute the expression for $\psi^{(2)} - \phi^{(2)}$ in eq. (5.4.5) to find

$$\psi^{(2)\prime\prime} + 3\mathcal{H}\psi^{(2)\prime} + a^2\Lambda\psi^{(2)} = \mathcal{H}Q' + a^2\Lambda Q + \nabla^{-2}\partial_i\partial^j + (\varphi')^2 - (\partial^i\varphi\partial_i\varphi)P^i_j , \tag{5.4.8}$$

where Q is defined by $\nabla^4 Q = -\nabla^2 P^i_i + 3\partial_i\partial^j P^i_j$.

Finally, we find the evolution equation for $\psi^{(2)}$:

$$\psi^{(2)\prime\prime} + 3\mathcal{H}\psi^{(2)\prime} + a^2\Lambda\psi^{(2)} = S(\eta) \tag{5.4.9}$$

where $S(\eta)$ is the source term in eq. (5.4.8). Employing the first order solution, eq. (5.4.4), it reads

$$\begin{aligned}
S(\eta) =& g^2\Omega_m\mathcal{H}^2\left[\frac{(f-1)^2}{\Omega_m}\varphi_0^2 + 2\left(2\frac{(f-1)^2}{\Omega_m} - \frac{3}{\Omega_m} + 3\right)\times\right. \\
&\left.\times\left(\nabla^{-2}(\partial^i\varphi_0\partial_i\varphi_0) - 3\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_0\partial_i\varphi_0)\right)\right] + \\
&+ g^2\left[\frac{4}{3}\left(\frac{f^2}{\Omega_m} + \frac{3}{2}\right)\nabla^{-2}\partial_i\partial^j(\partial^i\varphi_0\partial_i\varphi_0) - (\partial^i\varphi_0\partial_i\varphi_0)\right]
\end{aligned} \tag{5.4.10}$$

where the function $f(\eta)$ is defined by $f(\eta) = 1 + \frac{g'(\eta)}{\mathcal{H}g(\eta)}$.

The solution to eq. (5.4.9) is given by

$$\psi^{(2)}(\eta) = \frac{g}{g_m}\psi(\eta_m)^{(2)} + \psi_+^{(2)}(\eta)\int_{\eta_m}^{\eta} d\eta' \frac{\psi_-^{(2)}(\eta')}{W(\eta')}S(\eta') - \psi_-^{(2)}(\eta)\int_{\eta_m}^{\eta} d\eta' \frac{\psi_+^{(2)}(\eta')}{W(\eta')}S(\eta') \tag{5.4.11}$$

where $\psi_+^{(2)} = g(\eta)$ and $\psi_-^{(2)} = \frac{\mathcal{H}(\eta)}{a^2(\eta)}$ are respectively the growing and decaying solutions of the homogeneous equation, and W is the Wronskian $W(\eta) = \frac{1}{a^3(\eta)}\mathcal{H}_0^2(f_0 + \frac{3}{2}\Omega_{0m})$.

The initial conditions are represented by the term $\psi^{(2)}(\eta_m)$, which we choose to take in the matter dominated era: this will account for the primordial contribution to the non-Gaussianity in the perturbations.

The evolution of the gravitational potential $\phi^{(2)}$ is provided by the relation between $\psi^{(2)}$ and $\phi^{(2)}$:

$$\nabla^4\phi^{(2)} = \nabla^4\psi^{(2)} + 4g^2\nabla^4\varphi_0^2 - \frac{4}{3}g^2\left(\frac{f^2}{\Omega_m} + \frac{3}{2}\right)\left[\nabla^2(\partial^i\varphi_0\partial_i\varphi_0) - 3\partial_i\partial^j(\partial^i\varphi_0\partial_i\varphi_0)\right] . \tag{5.4.12}$$

We now discuss the important issue of initial conditions, which we fix at the time when the cosmological perturbations relevant today for the CMB anisotropies and LSS are well outside the horizon. We have seen that it is useful to consider the gauge-invariant quantity $\zeta \simeq \zeta^{(1)} + \frac{1}{2}\zeta^{(2)}$, because it is conserved on large scales for adiabatic perturbations and it carries all the information about the primordial level of non-Gaussianity. Different scenarios of generation of the cosmological perturbations are characterized by different values of $\zeta^{(2)}$, and it is standard use to parametrize the primordial level of non-Gaussianity through the parameter a_{NL}:

$$\zeta^{(2)} = 2a_{NL}(\zeta^{(1)})^2 . \tag{5.4.13}$$

At linear order during the matter dominated epoch and on large scales we have that $\zeta^{(1)} = -\frac{5}{3}\varphi(\eta_m)$, so we can write

$$\zeta^{(2)} = \frac{50}{9}a_{NL}\varphi_m^2 = \frac{50}{9}a_{NL}g^2(\eta_m)\varphi_0^2 . \tag{5.4.14}$$

On the other hand, by using the definition of $\zeta^{(2)}$ given in eq. (3.4.5) in the matter dominated epoch, the second-order $(0-0)$ component of the Einstein equations $-\mathcal{H}(\psi^{(2)'} + \mathcal{H}\phi^{(2)}) + 4\mathcal{H}^2\varphi_m^2 = \frac{8\pi G}{3}a^2\delta^{(2)}\rho_m$, we can express $\phi_m^{(2)}$ in terms of $\zeta^{(2)}$ as

$$\phi_m^{(2)} = -\frac{3}{5}\zeta^{(2)} + \frac{16}{3}\varphi_m^2 + 2\nabla^{-2}(\partial^i\varphi_m\partial_i\varphi_m) - 6\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_m\partial_j\varphi_m) . \tag{5.4.15}$$

In terms of the parameter a_{NL} and using eq. (5.4.12), the initial conditions read:

$$\phi^{(2)}_m = 2g^2_m \left[\left(-\frac{5}{3}(a_{NL}-1)+1\right)\varphi_0^2 + \nabla^{-2}(\partial^i\varphi_0\partial_i\varphi_0) - 3\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\right] , \tag{5.4.16}$$

$$\psi^{(2)}_m = 2g^2_m \left[\left(-\frac{5}{3}(a_{NL}-1)-1\right)\varphi_0^2 - \frac{2}{3}\nabla^{-2}(\partial^i\varphi_0\partial_i\varphi_0) - 3\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\right] . \tag{5.4.17}$$

After having determined the initial conditions, the solutions for $\psi^{(2)}$ and $\phi^{(2)}$ are

$$\begin{aligned}\psi^{(2)} =& 2g(\eta)g_m \left[\left(-\frac{5}{3}(a_{NL}-1)-1\right)\varphi_0^2 - \frac{2}{3}\nabla^{-2}(\partial^i\varphi_0\partial_i\varphi_0) - 3\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\right] + \\ &+\mathcal{H}_0^{-2}\left(f_0+\frac{3}{2}\Omega_{0m}\right)^{-1}\left[g(\eta)\int_{\eta_m}^{\eta} d\eta' a(\eta')\mathcal{H}(\eta')S(\eta') - \frac{\mathcal{H}(\eta)}{a^2(\eta)}\int_{\eta_m}^{\eta} d\eta' a^3(\eta')g(\eta')S(\eta')\right] ,\end{aligned} \tag{5.4.18}$$

$$\phi^{(2)} = \psi^{(2)} + 4g^2(\eta)\varphi_0^2 + \frac{4}{3}g^2(\eta)\left(\frac{f^2}{\Omega_m}+\frac{3}{2}\right)\left[\nabla^{-2}(\partial^i\varphi_0\partial_i\varphi_0) - 3\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\right] . \tag{5.4.19}$$

We can rewrite these expression in the following compact way:

$$\begin{aligned}\psi^{(2)} =& \left(B_1(\eta) - 2g(\eta)g_m - \frac{10}{3}(a_{nl}-1)g(\eta)g_m\right)\varphi_0^2 + \\ &+ \left(B_2(\eta) - \frac{4}{3}g(\eta)g_m\right)\left[\nabla^{-2}(\partial^i\varphi_0\partial_i\varphi_0) + -3\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\right] + ' \\ &+ B_3(\eta)\nabla^{-2}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0) + B_4(\eta)(\partial^i\varphi_0\partial_j\varphi_0) ,\end{aligned} \tag{5.4.20}$$

$$\begin{aligned}\phi^{(2)} =& \left(B_1(\eta) + 4g^2(\eta) - 2g(\eta)g_m - \frac{10}{3}(a_{nl}-1)g(\eta)g_m\right)\varphi_0^2 + \\ &+ \left[B_2(\eta) - \frac{4}{3}g(\eta)g_m + \frac{4}{3}g^2(\eta)\left(\frac{f^2}{\Omega_m}+\frac{3}{2}\right)\right]\times \\ &\times\left[\nabla^{-2}(\partial^i\varphi_0\partial_i\varphi_0) - 3\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\right] + \\ &+ B_3(\eta)\nabla^{-2}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0) + B_4(\eta)(\partial^i\varphi_0\partial_j\varphi_0) .\end{aligned} \tag{5.4.21}$$

where we have introduced the functions $B_i(\eta) = \mathcal{H}_0^{-2}\left(f_0+\frac{3}{2}\Omega_{0m}\right)\tilde{B}_i(\eta)$

with the following definitions:

$$\tilde{B}_1(\eta) = \int_{\eta_m}^{\eta} d\eta' \mathcal{H}^2(\eta')(f(\eta') - 1)^2 C(\eta, \eta') \,, \tag{5.4.22}$$

$$\tilde{B}_2(\eta) = 2\int_{\eta_m}^{\eta} d\eta' \mathcal{H}^2(\eta') \left[2(f(\eta') - 1)^2 - 3 + 3\Omega_m(\eta')\right] C(\eta, \eta') \,, \tag{5.4.23}$$

$$\tilde{B}_3(\eta) = \frac{4}{3}\int_{\eta_m}^{\eta} d\eta' \left(\frac{f^2(\eta')}{\Omega_m(\eta')} + \frac{3}{2}\right) C(\eta, \eta') \,, \quad \tilde{B}_4(\eta) = -\int_{\eta_m}^{\eta} d\eta' C(\eta, \eta') \,, \tag{5.4.24}$$

and

$$C(\eta, \eta') = g^2(\eta') a(\eta') \left[g(\eta)\mathcal{H}(\eta') - g(\eta')\frac{a^2(\eta')}{a^2(\eta)}\mathcal{H}(\eta)\right] \,. \tag{5.4.25}$$

In the expressions for the gravitational potentials, we recognize two contributions. There is a term which dominate on small scales and is negligible on large scales, that is $B_3(\eta)\nabla^{-2}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0) + B_4(\eta)(\partial^i\varphi_0\partial_j\varphi_0)$: this is insensitive to the non linearity in the initial conditions and gives rise to the Rees-Sciama effect due to the evolution in time of the second-order gravitational potentials. For large scales, the dominant contribution is given by the other terms, which carry the information on primordial non-Gaussianity and whose origin is purely relativistic. In an Einstein-de Sitter universe we would have $B_1(\eta) = B_2(\eta) = 0$ and there is no integrated contribution on large scales from these terms.

Temperature anisotropy

In order to compute the contribution to the temperature anisotropy on large scales given by the late ISW effect, we consider the expression of the integrated contributions to the temperature fluctuation at second order:

$$\begin{aligned}\frac{\delta^{(2)}T}{T} =& \frac{1}{2}\int_{\eta_\mathcal{E}}^{\eta_\mathcal{O}} d\eta \frac{\partial}{\partial\eta}\left[\phi^{(2)} + \psi^{(2)}\right] - I_1(\eta_\mathcal{E})\left(\varphi_\mathcal{E}^{(1)} + \tau^{(1)} - I_1(\eta_\mathcal{E})\right) + \\ &+ \int_{\eta_\mathcal{E}}^{\eta_\mathcal{O}} d\eta \left[4{k^{(1)}}^0\varphi' + 4\varphi'\varphi + 2{x^{(1)}}^0\varphi'' + 2{x^{(1)}}^i\varphi'_{,i}\right] \,.\end{aligned} \tag{5.4.26}$$

In this equation $_\mathcal{E}$ denotes a quantity evaluated at last scattering, I_1 is given by

$$I_1(\eta) = 2\int_{\eta_\mathcal{O}}^{\eta} d\tilde{\eta}\varphi' \,, \tag{5.4.27}$$

while ${k^{(1)}}^0$ and ${x^{(1)}}^\mu$ are the first order perturbations to the background wavevector and geodesic, respectively, and are given by

$$k^{(1)^0}(\eta) =- 2\varphi + I_1(\eta) \tag{5.4.28}$$

$$x^{(1)^0}(\eta) =2\int_{\eta_{\mathcal{O}}}^{\eta} d\tilde{\eta}\left[-\varphi + (\eta-\tilde{\eta})\varphi'\right] \tag{5.4.29}$$

$$x^{(1)^i}(\eta) =- 2\int_{\eta_{\mathcal{O}}}^{\eta} d\tilde{\eta}\left[\varphi e^i + (\eta-\tilde{\eta})\varphi^{,i}\right] , \tag{5.4.30}$$

where e^i is the unit vector specifying the line of sight direction. All the integrals are to be evaluated along the background geodesic $x^{(0)^\mu} = (\eta, \mathbf{\hat{e}}(\eta_{\mathcal{O}} - \eta))$; for example, in eq. (5.4.27) we have $\varphi' \equiv \varphi'(\tilde{\eta}, \mathbf{x} = \mathbf{\hat{e}}(\eta_{\mathcal{O}} - \tilde{\eta}))$.

Now we notice that eq. (5.4.26) can be simplified. To this end, we use the relations

$$x^{(1)^0}(\eta) + x^{(1)^i}(\eta)e_i = -2\int_{\eta_{\mathcal{O}}}^{\eta} d\tilde{\eta}\varphi' \tag{5.4.31}$$

$$2\int_{\eta_{\mathcal{O}}}^{\eta} d\tilde{\eta}\varphi' I_1(\tilde{\eta}) = \frac{1}{2}[I_1(\eta)]^2 \tag{5.4.32}$$

and the decomposition of the total derivative along the background geodesic for a generic function $f(\eta, x^i(\eta))$, $\frac{df}{d\eta} = f' - e^i f_{,i}$, to obtain

$$\begin{aligned}\frac{\delta^{(2)}T}{T} =& \frac{1}{2}\int_{\eta_{\mathcal{E}}}^{\eta_{\mathcal{O}}} d\eta \frac{\partial}{\partial\eta}\left[\phi^{(2)} + \psi^{(2)}\right] - I_1(\eta_{\mathcal{E}})\left({\varphi^{(1)}}_{\mathcal{E}} + \tau^{(1)}\right) + \frac{1}{2}[I_1(\eta_{\mathcal{E}})]^2 + \\ &- 4\int_{\eta_{\mathcal{E}}}^{\eta_{\mathcal{O}}} d\eta\left[\varphi\varphi' + \varphi''\int_{\eta_{\mathcal{O}}}^{\eta} d\tilde{\eta}\varphi\right] + 4\varphi'_{\mathcal{E}}\int_{\eta_{\mathcal{E}}}^{\eta_{\mathcal{O}}} d\eta(\eta_{\mathcal{E}} - \eta)\varphi\,.\end{aligned} \tag{5.4.33}$$

As we have already noted, the second order ISW effect is separated into two parts: the early ISW effect due to a non negligible radiation component at the time of last scattering, and the late ISW effect due to expansion growth suppression during the Λ-dominated epoch. Therefore we split every integral in eq. (5.4.33) as $\int_{\eta_{\mathcal{E}}}^{\eta_{\mathcal{O}}} d\eta = \int_{\eta_{\mathcal{E}}}^{\eta_m} d\eta + \int_{\eta_m}^{\eta_{\mathcal{O}}} d\eta$, where η_m represents the epoch when full matter domination is reached.

For the late ISW effect, we need to take only the latter contribution. Using the expression for the intrinsic temperature anisotropy on large scales $\tau = -\frac{2}{3}\varphi_{\mathcal{E}}$ and the solutions for the potentials in eqs. (5.4.20-5.4.21), we finally find

$$\begin{aligned}\frac{1}{2}\frac{\delta^{(2)}T}{T}^{\text{late}} =& \int_{\eta_m}^{\eta_{\mathcal{O}}} d\eta\left[2\left(-1 - \frac{5}{3}(a_{NL} - 1)\right)g_m g'(\eta) + B_1'(\eta) + 4g(\eta)g'(\eta)\right]\varphi_0^2\big|_{\mathbf{x}=\mathbf{\hat{e}}(\eta_{\mathcal{O}}-\eta)} + \\ &+ \int_{\eta_m}^{\eta_{\mathcal{O}}} d\eta g_m^2 \bar{B}(\eta)\left[\nabla^{-2}(\partial^i\varphi_0\partial_i\varphi_0) - 3\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_0\partial_j\varphi_0)\right]\big|_{\mathbf{x}=\mathbf{\hat{e}}(\eta_{\mathcal{O}}-\eta)} + \\ &- 4\int_{\eta_m}^{\eta_{\mathcal{O}}} d\eta\left[\varphi\varphi' + \varphi''\int_{\eta_{\mathcal{O}}}^{\eta} d\tilde{\eta}\varphi\right] + \frac{1}{2}[I_1(\eta_m)]^2 - \frac{1}{3}\varphi_\eta I_1(\eta_m)\end{aligned} \tag{5.4.34}$$

where for simplicity of notation we have introduced the function

$$\bar{B}(\eta) = \left(\frac{B_2'(\eta)}{g_m^2} - \frac{4}{3}\frac{g'(\eta)}{g_m}\right) + \frac{4}{3}\frac{g'(\eta)g(\eta)}{g_m^2}\left(e(\eta) + \frac{3}{2}\right) + \frac{2}{3}\frac{g^2(\eta)}{g_m^2}e'(\eta) \tag{5.4.35}$$

and the first order potentials are given by $\varphi(\eta, \mathbf{x}) = g(\eta)\varphi_0(\mathbf{x})$.

A simple but important comment about eq. (5.4.34) is that the whole effect is vanishing in the case of a vanishing dark-energy component, since it arises from the explicit time dependence of the linear gravitational potentials during the late accelerated phase. Moreover, we can recognize two different contributions. The first, proportional to $[-\frac{5}{3}(a_{NL} - 1)]$, is directly related to the primordial level of non-Gaussianity: in the case of strong non-Gaussianity, $a_{NL} \gg 1$, the late ISW effect on large scales can be strongly amplified and this is the dominant contribution. The remaining terms are due to the non-linear evolution of the gravitational potentials and to the second-order corrections to the temperature anisotropies.

Angular decomposition

In order to provide the expression for the angular bispectrum we expand the temperature anisotropy in spherical harmonics obtaining the multipoles a_{lm}:

$$\frac{\delta T(\hat{\mathbf{n}})}{T} = \sum_{lm} a_{lm} Y(\hat{\mathbf{n}})_{lm} \Longrightarrow a_{lm} = \int d^2\hat{\mathbf{n}} \frac{\delta T(\hat{\mathbf{n}})}{T} Y^*_{lm}(\hat{\mathbf{n}})\,. \tag{5.4.36}$$

Since the temperature anisotropy is the sum of a linear part and a non linear one, we decompose the multipoles a_{lm} into a linear (Gaussian) part a^L_{lm} and a non linear (non-Gaussian) contribution a^{NL}_{lm}:

$$a_{lm} = a^L_{lm} + a^{NL}_{lm}\,. \tag{5.4.37}$$

The linear multipole is expressed as

$$a^L_{lm} = \int d^2\hat{\mathbf{n}} \frac{\delta^{(1)}T(\hat{\mathbf{n}})}{T} Y^*_{lm}(\hat{\mathbf{n}}) = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} {\phi^{(1)}}_i(\mathbf{k}) {\Delta^{(1)}}_l(k) Y^*_{lm}(\hat{\mathbf{k}}) \tag{5.4.38}$$

where ${\Delta^{(1)}}_l(k)$ is the linear radiation transfer function which gives the relation between the initial fluctuations ${\phi^{(1)}}_i(\mathbf{k})$ and the observed temperature anisotropies. For the linear Sachs-Wolfe effect on large scales (small l), the transfer function is ${\Delta^{(1)}}_l(k) = \frac{1}{3}j_l(k(\eta_0 - \eta_\mathcal{E}))$, where j_l are the spherical Bessel functions of order l.

Expressing in a similar way the non linear multipoles a^{NL}_{lm} in terms of the initial potential fluctuations corresponds to obtain a second order radiation transfer function. With our results for the temperature anisotropies on large scales we are able to compute it for small l.

Now we derive a_{lm}^{NL} for the late ISW effect. Let us consider the contribution arising from the time derivatives of the second order gravitational potentials in eq. (5.4.34), and write it in terms of its Fourier transform:

$$\frac{1}{2}\frac{\delta^{(2)}T}{T}[\psi^{(2)\prime}+\phi^{(2)\prime}]=\int\frac{d^3k}{(2\pi)^3}\int_{\eta_m}^{\eta_{\mathcal{O}}}d\eta\left[\left(\frac{10}{3}(a_{NL}-1)g_m g'(\eta)-2g_m g'(\eta)+\right.\right.$$
$$\left.\left.+B_1'(\eta)+4g(\eta)g'(\eta)\right)[\varphi_0^2](\mathbf{k})-g_m^2\bar{B}(\eta)K_2(\mathbf{k})\right]e^{-i\mathbf{k}\cdot\hat{\mathbf{n}}(\eta_{\mathcal{O}}-\eta)} \tag{5.4.39}$$

where $[\varphi_0^2](\mathbf{k})$ is the convolution giving the Fourier transform of $\varphi_0^2(\mathbf{x})$, and $K_2(\mathbf{k})$ is defined as

$$K_2(\mathbf{k})=\frac{1}{(2\pi)^3}\int d^3k_1 d^3k_2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\left(3\frac{(\mathbf{k}_1\cdot\mathbf{k})(\mathbf{k}_2\cdot\mathbf{k})}{k^4}-\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k^2}\right)\varphi_m(\mathbf{k}_1)\varphi_m(\mathbf{k}_2)\,. \tag{5.4.40}$$

We now make use of the Rayleigh expansion

$$e^{-i\mathbf{k}\cdot\mathbf{x}}=4\pi\sum_{l,m}(-i)^l j_l(kx)Y_{lm}^*(\hat{\mathbf{k}})Y_{lm}(\hat{\mathbf{x}}) \tag{5.4.41}$$

and, using the orthonormality of the spherical harmonics, we find

$$a_{lm}^{NL}[\psi^{(2)\prime}+\phi^{(2)\prime}]=4\pi(-i)^l\int\frac{d^3k}{(2\pi)^3}\int_{\eta_m}^{\eta_{\mathcal{O}}}d\eta j_l(k(\eta_0-\eta))\times$$
$$\times\left[\left(\frac{10}{3}(a_{NL}-1)\frac{g'(\eta)}{g_m}-2\frac{g'(\eta)}{g_m}+\frac{B_1'(\eta)}{g_m^2}+4\frac{g(\eta)g'(\eta)}{g_m^2}\right)[\varphi_m^2](\mathbf{k})-\bar{B}(\eta)K_2(\mathbf{k})\right] \tag{5.4.42}$$

where we have switched to φ_m for the initial conditions.

In the remaining terms, which are additional second order corrections built up from the first order potential, we have a different dependence on $\hat{\mathbf{n}}$. For example, if we consider the integral over φ'' in eq. (5.4.34) we have

$$-4\int_{\eta_m}^{\eta_{\mathcal{O}}}d\eta\left[\int_{\eta_{\mathcal{O}}}^{\eta}d\tilde{\eta}\varphi\right]g''(\eta)\,\varphi_0(\mathbf{x})|_{\mathbf{x}=-\hat{\mathbf{n}}(\eta_{\mathcal{O}}-\eta)}=$$
$$=-4\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\int_{\eta_m}^{\eta_{\mathcal{O}}}d\eta\frac{g''(\eta)}{g_m}\int_{\eta_{\mathcal{O}}}^{\eta}d\tilde{\eta}\frac{g(\tilde{\eta})}{g_m}\varphi_m(\mathbf{k}_1)\varphi_m(\mathbf{k}_2)e^{-i\mathbf{k}_1\cdot\hat{\mathbf{n}}(\eta_{\mathcal{O}}-\eta)}e^{-i\mathbf{k}_2\cdot\hat{\mathbf{n}}(\eta_{\mathcal{O}}-\tilde{\eta})}=$$
$$=-4(4\pi)^2\sum_{L_1,M_1}\sum_{L_2,M_2}(-i)^{L_1+L_2}Y_{L_1M_1}^*(\hat{\mathbf{n}})Y_{L_2M_2}^*(\hat{\mathbf{n}})\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\times$$
$$\times\int_{\eta_m}^{\eta_{\mathcal{O}}}d\eta\frac{g''(\eta)}{g_m}j_{L_1}(k_1(\eta_{\mathcal{O}}-\eta))\int_{\eta_{\mathcal{O}}}^{\eta}d\tilde{\eta}\frac{g(\tilde{\eta})}{g_m}j_{L_2}(k_2(\eta_{\mathcal{O}}-\tilde{\eta}))\times$$
$$\times\varphi_m(\mathbf{k}_1)\varphi_m(\mathbf{k}_2)Y_{L_1M_1}(\hat{\mathbf{k}}_1)Y_{L_2M_2}(\hat{\mathbf{k}}_2) \tag{5.4.43}$$

and a similar expression holds for the other terms.

In general, we can write the following expression for a_{lm}^{NL}:

$$a_{lm}^{NL} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \left[K_0(\mathbf{k}){\Delta^{(2)}}_l^0(k) + K_1(\mathbf{k}){\Delta^{(2)}}_l^1(k) + K_3(\mathbf{k}){\Delta^{(2)}}_l^2(k)\right] Y_{lm}^*(\hat{\mathbf{k}})+ + (4\pi)^2 \sum_{L_1,M_1} \sum_{L_2,M_2} (-i)^{L_1+L_2} \mathcal{G}_{lL_1L_2}^{mM_1M_2} \times \times \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \varphi_m(\mathbf{k}_1)\varphi_m(\mathbf{k}_2)\Delta_{L_1L_2}(k_1,k_2)Y_{L_1M_1}(\hat{\mathbf{k}}_1)Y_{L_2M_2}(\hat{\mathbf{k}}_2) \tag{5.4.44}$$

where $K_n(\mathbf{k})$ are convolutions in Fourier space expressed in terms of some kernels $f_n(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k})$ as

$$K_n(\mathbf{k}) = \frac{1}{(2\pi)^3} \int d^3k_1 d^3k_2 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) f_n(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k})\varphi_m(\mathbf{k}_1)\varphi_m(\mathbf{k}_2) \tag{5.4.45}$$

with

$$f_0(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) = -\frac{5}{3}(a_{NL} - 1) - 1\,, \quad f_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) = 1 \tag{5.4.46}$$

$$f_2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}) = 3\frac{(\mathbf{k}_1 \cdot \mathbf{k})(\mathbf{k}_2 \cdot \mathbf{k})}{k^4} - \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k^2} \tag{5.4.47}$$

and, correspondingly,

$${\Delta^{(2)}}_l^0(k) = 2\int_{\eta_m}^{\eta_\mathcal{O}} d\eta \frac{g'(\eta)}{g_m} j_l(k(\eta_0 - \eta))\,, \tag{5.4.48}$$

$${\Delta^{(2)}}_l^0(k) = \int_{\eta_m}^{\eta_\mathcal{O}} d\eta \frac{B_1'(\eta)}{g_m^2} j_l(k(\eta_0 - \eta))\,, \tag{5.4.49}$$

$${\Delta^{(2)}}_l^0(k) = -\int_{\eta_m}^{\eta_\mathcal{O}} d\eta \bar{B}(\eta) j_l(k(\eta_0 - \eta))\,. \tag{5.4.50}$$

Moreover, in eq. (5.4.44), $\mathcal{G}_{lL_1L_2}^{mM_1M_2} = \int d^2\hat{\mathbf{n}} Y_{lm}(\hat{\mathbf{n}})Y_{L_1M_1}(\hat{\mathbf{n}})Y_{L_2M_2}(\hat{\mathbf{n}})$ is the Gaunt integral and

$$\Delta_{L_1L_2}(k_1, k_2) = \int_{\eta_m}^{\eta_\mathcal{O}} d\eta \frac{g''(\eta)}{g_m} j_{L_1}(k_1(\eta_\mathcal{O} - \eta)) \int_{\eta_\mathcal{O}}^{\eta} d\tilde{\eta} \frac{g(\tilde{\eta})}{g_m} j_{L_2}(k_2(\eta_\mathcal{O} - \tilde{\eta}))+ + 2\int_{\eta_m}^{\eta_\mathcal{O}} d\eta \frac{g'(\eta)}{g_m} j_{L_1}(k_1(\eta_\mathcal{O} - \eta)) \times \times \left[2\int_{\eta_m}^{\eta_\mathcal{O}} d\eta \frac{g'(\eta)}{g_m} j_{L_2}(k_2(\eta_\mathcal{O} - \eta)) + \frac{1}{3} j_{L_2}(k_2(\eta_\mathcal{O} - \eta_\mathcal{E}))\right]\,. \tag{5.4.51}$$

We see that eq. (5.4.44) is the generalization of the linear relation (5.4.38), with the functions ${\Delta^{(2)}}_l^n(k)$ and $\Delta_{L_1L_2}(k_1, k_2)$ playing the role of coefficients of the second order transfer function, which relates the quadratic curvature perturbations to the observed temperature anisotropies, and for the second order late ISW effect these depend on the growth suppression factor $g(\eta)$.

5.4.2 Early ISW effect

Evolution of the gravitational potentials

At last scattering, the universe is not completely matter dominated, and the residual radiation component make the potentials decay in time. This gives rise to the so-called early ISW effect, which gives a non neglgible contribution to the large-scale CMB anisotropies since it almost mimics a Sachs-Wolfe effect: for example, at the linear level this effect gives a correction to the power spectrum normalization of about 20%. From the expression (5.4.33) we expect that also at second order the early integrated effect gives a significant correction to the second order Sachs-Wolfe effect. So we now derive the evolution of the gravitational potentials for a universe filled by pressureless matter with energy density ρ_m and $w_m = 0$, and radiation with energy density ρ_γ and $w_m = \frac{1}{3}$, neglecting the cosmological constant term at last scattering. At first order, the linear growing mode for the gravitational potential $\varphi = \phi^{(1)} = \psi^{(1)}$ is

$$\varphi(\eta) = \frac{F(\eta)}{F(\eta_\mathcal{E})}\varphi_\mathcal{E} \tag{5.4.52}$$

where the function F is

$$F(\eta) = 1 - \frac{H}{a}\int_{a_i}^{a}\frac{d\tilde{a}}{H} = \frac{3}{5} + \frac{2}{15y} - \frac{8}{15y^2} - \frac{16}{15y^3} + \frac{16\sqrt{1+y}}{15y^3} \tag{5.4.53}$$

with $y \equiv \frac{\rho_m}{\rho_\gamma} \propto a$, and a_i is some epoch in the radiation-dominated period.

At second order, we find from the traceless $(i-j)$ Einstein equation a relation between $\psi^{(2)}$ and $\phi^{(2)}$:

$$\phi^{(2)} = \psi^{(2)} + 4\varphi^2 - Q \tag{5.4.54}$$

where Q can be written as

$$Q = -\frac{3}{F_\mathcal{E}^2}\tilde{Q}\left[\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_\mathcal{E}\partial_j\varphi_\mathcal{E}) - \frac{1}{3}\nabla^{-2}(\partial^i\varphi_\mathcal{E}\partial_i\varphi_\mathcal{E})\right] \tag{5.4.55}$$

where $\tilde{Q}$ is given by

$$\tilde{Q} = -4\frac{1+y}{4+3y}\left[\dot{F}^2 + 2F\dot{F} + \frac{1}{2}\frac{6+5y}{1+y}F^2\right] \tag{5.4.56}$$

Now we will derive the evolution equation for the potential $\psi^{(2)}$. We start from the expression for $\zeta^{(2)}$ in the Poisson gauge

$$\zeta^{(2)} = -\psi^{(2)} - \mathcal{H}\frac{\delta^{(2)}\rho}{\rho'} + \Delta\zeta^{(2)} \tag{5.4.57}$$

where the last term is given by products of first order perturbations,

$$\Delta\zeta^{(2)} = 2\mathcal{H}\frac{\delta^{(1)}\rho'}{\rho'}\frac{\delta^{(1)}\rho}{\rho'} + 2\frac{\delta^{(1)}\rho}{\rho'}(\varphi' + 2\mathcal{H}\varphi) - \left(\frac{\delta^{(1)}\rho}{\rho'}\right)^2\left(\mathcal{H}\frac{\rho''}{\rho} - \mathcal{H}' - 2\mathcal{H}^2\right). \tag{5.4.58}$$

Using the second order $(0-0)$ component of the Einstein equations, we express $\delta^{(2)}\rho$ in terms of the gravitational potentials, to find

$$\zeta^{(2)} = -\psi^{(2)} + 2\frac{\rho}{\rho'}\left(\psi^{(2)\prime} + \mathcal{H}\phi^{(2)}\right) - 8\frac{\rho}{\rho'}\mathcal{H}\varphi^2 - 2\frac{\rho}{\mathcal{H}\rho'}(\varphi')^2\Delta\zeta^{(2)}. \tag{5.4.59}$$

Finally, using the relation between the gravitational potentials to eliminate $\phi^{(2)}$ we get

$$\zeta^{(2)} = -\psi^{(2)} + 2\frac{\rho}{\rho'}\left(\psi^{(2)\prime} + \mathcal{H}\psi^{(2)}\right) - 8\frac{\rho}{\rho'}\mathcal{H}Q - 2\frac{\rho}{\mathcal{H}\rho'}(\varphi')^2\Delta\zeta^{(2)}, \tag{5.4.60}$$

which can be rewritten as

$$\frac{\sqrt{\rho}}{a\mathcal{H}}\left[\frac{a}{\sqrt{\rho}}\psi^{(2)}\right]' = \frac{1}{2}\frac{\rho'}{\mathcal{H}\rho}\zeta^{(2)} + \frac{\varphi'^2}{\mathcal{H}^2} + Q - \frac{1}{2}\frac{\rho'}{\mathcal{H}\rho}\Delta\zeta^{(2)}. \tag{5.4.61}$$

We have to find an expression for $\Delta\zeta^{(2)}$. To this end, from the background continuity equations for matter and radiation we find for the total energy density

$$\frac{\rho''}{\rho'} = \frac{\mathcal{H}'}{\mathcal{H}} - \frac{9\rho_m + 16\rho_\gamma}{3\rho_m + 4\rho_\gamma}, \tag{5.4.62}$$

hence

$$\mathcal{H}\frac{\rho''}{\rho} - \mathcal{H}' - 2\mathcal{H}^2 = -\mathcal{H}^2\left(6 - \frac{3\rho_m}{3\rho_m + 4\rho_\gamma}\right). \tag{5.4.63}$$

Using the total energy continuity equation at first order

$$\delta\rho'^{(1)} = -3\mathcal{H}\left(\delta^{(1)}\rho + \frac{1}{3}\delta^{(1)}\rho_\gamma\right) - 3\left(\rho + \frac{1}{3}\rho_\gamma\right)\varphi' \tag{5.4.64}$$

and the adiabaticity condition, we have

$$\frac{\delta\rho'^{(1)}}{\rho'} = -3\mathcal{H}\left(1 + \frac{4\rho_\gamma}{3\rho_m + 4\rho_\gamma}\right)\frac{\delta\rho^{(1)}}{\rho'} - \frac{\varphi'}{\mathcal{H}}. \tag{5.4.65}$$

Finally, we use the $(0-0)$ first order Einstein equation, $\varphi' + \mathcal{H}\varphi = -\frac{1}{2}\mathcal{H}\frac{\delta^{(1)}\rho}{\rho}$, to get

$$\Delta\zeta^{(2)} = \left[2\frac{\mathcal{H}\rho}{\rho'}\left(\frac{\varphi'}{\mathcal{H}} - \varphi\right)\right]^2 \left(1 + \frac{4\rho_\gamma}{3\rho_m + 4\rho_\gamma}\right) - 8\frac{\mathcal{H}\rho}{\rho'}\varphi\left(\frac{\varphi'}{\mathcal{H}} - \varphi\right). \tag{5.4.66}$$

Now, using the linear growing mode, we rewrite it in the form

$$\frac{1}{2}\frac{\rho'}{\mathcal{H}\rho}\Delta\zeta^{(2)} = \frac{1}{F_{\mathcal{E}}^2}\left[2R\frac{F'^2}{\mathcal{H}^2} - 2(2-R)F^2 - 4(1-R)\frac{FF'}{\mathcal{H}}\right]\varphi_{\mathcal{E}}^2 \tag{5.4.67}$$

where for simplicity we have introduced the function

$$R(y) = \frac{1+y}{4+3y}\left(1 + \frac{4}{4+3y}\right). \tag{5.4.68}$$

In the case of adiabatic perturbations, when $\zeta^{(2)}$ is constant, the integration of eq. (5.4.61) yields

$$\psi^{(2)} = -F(\eta)\zeta^{(2)} + \frac{\sqrt{\rho}}{a}\int_{a_i}^{a}\frac{da}{\sqrt{\rho}}\left[\frac{\varphi'^2}{\mathcal{H}^2} + Q - \frac{\rho'}{\mathcal{H}\rho}\Delta\zeta^{(2)}\right] + C\frac{\sqrt{\rho}}{a}, \tag{5.4.69}$$

and by using eq. (5.4.54) we have

$$\begin{aligned}\psi^{(2)} + \phi^{(2)} =& -2F(\eta)\zeta^{(2)} + \frac{1}{F_{\mathcal{E}}^2}\left\{4F^2 + \right.\\ &\left. + 2\frac{\sqrt{\rho}}{a}\int_{a_i}^{a}\frac{da}{\sqrt{\rho}}\left[(1-2R)\frac{(F')^2}{\mathcal{H}^2} + 2(2-R)F^2 + 4(1-R)\frac{FF'}{\mathcal{H}}\right]\right\}\varphi_{\mathcal{E}}^2 + \\ & + \frac{3}{10F_{\mathcal{E}}^2}\left[\tilde{Q} - 2\frac{\sqrt{\rho}}{a}\int_{a_i}^{a}\frac{da}{\sqrt{\rho}}\tilde{Q}\right]\mathcal{K},\end{aligned} \tag{5.4.70}$$

which is the expression we need for the second order early ISW effect.

Temperature anisotropies and angular decomposition

The expression for the second order early ISW effect reads

$$\begin{aligned}\frac{\delta^{(2)}T}{T} =& \frac{1}{2}\int_{\eta_{\mathcal{E}}}^{\eta_m} d\eta\, \frac{\partial}{\partial\eta}\left(\psi^{(2)} + \phi^{(2)}\right)(\mathbf{x},\eta)\Big|_{\mathbf{x}=\hat{\mathbf{e}}(\eta_{\mathcal{O}}-\eta)} + \\ & - 4\int_{\eta_{\mathcal{E}}}^{\eta_m} d\eta\left[\varphi\varphi' + \varphi''\int_{\eta_{\mathcal{O}}}^{\eta} d\tilde{\varphi}\varphi\right] + \frac{1}{2}I_1^2(\eta_m;\eta_{\mathcal{E}}) + \\ & + I_1(\eta_m;\eta_{\mathcal{E}})I_1(\eta_{\mathcal{E}}) - \frac{1}{3}\varphi_{\mathcal{E}}I_1^2(\eta_m;\eta_{\mathcal{E}}) - 4\varphi'_{\mathcal{E}}\int_{\eta_{\mathcal{E}}}^{\eta_{\mathcal{O}}} d\eta(\eta_{\mathcal{E}} - \eta)\varphi\end{aligned} \tag{5.4.71}$$

where we have splitted $I_1(\eta_\mathcal{E})$ as

$$I_1(\eta_\mathcal{E}) = -2\int_{\eta_\mathcal{E}}^{\eta_\mathcal{O}} d\eta\varphi' = -2\int_{\eta_\mathcal{E}}^{\eta_m} d\eta\varphi' - 2\int_{\eta_m}^{\eta_\mathcal{O}} d\eta\varphi' \equiv I_1(\eta_m;\eta_\mathcal{E}) + I_1(\eta_m)\,. \tag{5.4.72}$$

Note in particular that the last term in eq. (5.4.71) is proportional to $\varphi'_\mathcal{E}$ and would vanish in the limit of full matter domination at the recombination epoch.

The multipoles for the linear early ISW effect are

$$\begin{aligned} a^L_{lm} =& 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3}\left[2\int_{\eta_\mathcal{E}}^{\eta_m} d\eta\varphi'(\mathbf{k},\eta) j_l(k(\eta_\mathcal{O}-\eta))\right] Y^*_{lm}(\hat{\mathbf{k}}) \simeq \\ \simeq& 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3}\left[2\;\varphi(\mathbf{k},\eta)|^{\eta_m}_{\eta_\mathcal{E}}\, j_l(k(\eta_\mathcal{O}-\eta_\mathcal{E}))\right] Y^*_{lm}(\hat{\mathbf{k}}) = \\ =& 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3}\left[2\varphi_\mathcal{E}\frac{\Delta F}{F_\mathcal{E}} j_l(k(\eta_\mathcal{O}-\eta_\mathcal{E}))\right] Y^*_{lm}(\hat{\mathbf{k}}) \end{aligned} \tag{5.4.73}$$

where $\Delta F = F(\eta)|^{\eta_m}_{\eta_\mathcal{E}}$, and we have made the standard approximation of evaluating the Bessel function at $\eta_\mathcal{E}$, since most of the contribution to the early ISW effect comes from near recombination.

At second order we have in a similar way

$$a^{NL}_{lm}[\psi^{(2)\prime}+\phi^{(2)\prime}] = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}\left(\psi^{(2)}+\phi^{(2)}\right)\Big|^{\eta_m}_{\eta_\mathcal{E}} j_l(k(\eta_\mathcal{O}-\eta_\mathcal{E}))\right] Y^*_{lm}(\hat{\mathbf{k}}) \tag{5.4.74}$$

with the gravitational potentials given by eq. (5.4.70).

Explicitly, adding the contribution quadratic in the first order terms, and using the relation $\varphi_m = \frac{F_m}{F_\mathcal{E}}\varphi_\mathcal{E}$, we can write the non linear multipoles as

$$\begin{aligned} a^{NL}_{lm} =& 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3}\left[K_0(\mathbf{k}){\Delta^{(2)}}^0_l(k) + K_1(\mathbf{k}){\Delta^{(2)}}^1_l(k) + K_2(\mathbf{k}){\Delta^{(2)}}^2_l(k)\right] Y^*_{lm}(\hat{\mathbf{k}}) + \\ &+ (4\pi)^2 \sum_{L_1,M_1}\sum_{L_2,M_2} (-i)^{L_1+L_2} \mathcal{G}^{mM_1M_2}_{lL_1L_2} \times \\ \times& \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} \varphi_\mathcal{E}(\mathbf{k}_1)\varphi_\mathcal{E}(\mathbf{k}_2)\Delta_{L_1L_2}(k_1,k_2) Y_{L_1M_1}(\hat{\mathbf{k}}_1) Y_{L_2M_2}(\hat{\mathbf{k}}_2) \end{aligned} \tag{5.4.75}$$

where $\varphi_\mathcal{E}$ is the gravitational potential at last scattering. The convolutions $K_n(\mathbf{k})$ are given by

$$K_n(\mathbf{k}) = \frac{1}{(2\pi)^3}\int d^3k_1 d^3k_2 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}) f_n(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})\varphi_\mathcal{E}(\mathbf{k}_1)\varphi_\mathcal{E}(\mathbf{k}_2) \tag{5.4.76}$$

with the kernels defines as in eqs. (5.4.46-5.4.47), and the coefficients of the transfer function are given by

$$\Delta_l^{0(2)}(k) = \frac{6}{5}\frac{\Delta F}{F_{\mathcal{E}}^2} j_l(k(\eta_{\mathcal{O}} - \eta_{\mathcal{E}}))\,, \tag{5.4.77}$$

$$\Delta_l^{1(2)}(k) = \frac{1}{F_{\mathcal{E}}^2}\left\{\frac{\sqrt{\rho}}{a}\int_{a_i}^{a}\frac{da}{\sqrt{\rho}}\left[(1-2R)\frac{F'^2}{\mathcal{H}^2} + 2(2-R)F^2 + 4(1-R)\frac{FF'}{\mathcal{H}}\right]\right\}\Bigg|_{a_{\mathcal{E}}}^{a_m} \times \\ \times j_l(k(\eta_{\mathcal{O}} - \eta_{\mathcal{E}})) + \frac{4}{5}\frac{\Delta F}{F_{\mathcal{E}}^2} j_l(k(\eta_{\mathcal{O}} - \eta_{\mathcal{E}}))\,, \tag{5.4.78}$$

$$\Delta_l^{2(2)}(k) = \frac{1}{F_{\mathcal{E}}^2}\left[\frac{\tilde{Q}}{2} - \frac{\sqrt{\rho}}{a}\int_{a_i}^{a}\frac{da}{\sqrt{\rho}}\tilde{Q}\right]\Bigg|_{a_{\mathcal{E}}}^{a_m} j_l(k(\eta_{\mathcal{O}} - \eta_{\mathcal{E}}))\,, \tag{5.4.79}$$

and

$$\Delta_{L_1L_2}(k_1,k_2) = \left(2\frac{\Delta F^2}{F_{\mathcal{E}}^2} + \frac{2}{3}\frac{\Delta F}{F_{\mathcal{E}}}\right) j_{L_1}(k(\eta_{\mathcal{O}} - \eta_{\mathcal{E}}))j_{L_2}(k(\eta_{\mathcal{O}} - \eta_{\mathcal{E}})) + \\ +4\frac{\Delta F}{F_{\mathcal{E}}} j_{L_2}(k(\eta_{\mathcal{O}} - \eta_{\mathcal{E}}))\int_{\eta_m}^{\eta_{\mathcal{O}}} d\eta c\frac{g'(\eta)}{g_m} j_{L_1}(k(\eta_{\mathcal{O}} - \eta)) - \frac{F'(\eta_{\mathcal{E}})}{F_{\mathcal{E}}} j_{L_2}(k(\eta_{\mathcal{O}} - \eta))\times \\ \times\left[\int_{\eta_{\mathcal{E}}}^{\eta_m} d\eta(\eta_{\mathcal{E}} - \eta)\frac{F(\eta)}{F_{\mathcal{E}}} j_{L_1}(k(\eta_{\mathcal{O}} - \eta)) + \int_{\eta_m}^{\eta_{\mathcal{O}}} d\eta(\eta_{\mathcal{E}} - \eta)c\frac{g(\eta)}{g_m} j_{L_1}(k(\eta_{\mathcal{O}} - \eta))\right] + \\ + 4\int_{\eta_{\mathcal{E}}}^{\eta_m} d\eta\frac{F''(\eta)}{F_{\mathcal{E}}} j_{L_2}(k(\eta_{\mathcal{O}} - \eta))\times \\ \times\left[\int_{\eta}^{\eta_m} d\tilde{\eta}\frac{F(\tilde{\eta})}{F_{\mathcal{E}}} j_{L_1}(k(\eta_{\mathcal{O}} - \tilde{\eta})) + \int_{\eta_m}^{\eta_{\mathcal{O}}} d\tilde{\eta}c\frac{g(\tilde{\eta})}{g_m} j_{L_1}(k(\eta_{\mathcal{O}} - \tilde{\eta}))\right]\,, \tag{5.4.80}$$

where $c = \frac{F_m}{F_{\mathcal{E}}}$. From this equations it is apparent that also at second order the early ISW effect can be approximated on large scales as a Sachs-Wolfe effect, whose transfer function is proportional to $j_l(k(\eta_{\mathcal{O}} - \eta_{\mathcal{E}}))$.

5.4.3 Second order tensor perturbations

At second order, we expect that tensor modes give a non negligible contribution to the large-scale CMB anisotropies, since in single-field inflationary models their level is comparable to that of second order scalar perturbations.

The contribution to the CMB anisotropies is given by the integrated effect

$$\frac{\delta^{(2)}T}{T} = -\frac{1}{2}\int_{\eta_{\mathcal{E}}}^{\eta_{\mathcal{O}}} d\eta\, \chi_{ij}^{(2)\prime}(\eta,\mathbf{x})e^ie^j\Big|_{\mathbf{x}=\hat{\mathbf{e}}(\eta_{\mathcal{O}}-\eta)}\,. \tag{5.4.81}$$

On large scales the tensor modes remain constant, while they start to evolve when they are reentering the horizon. Therefore the main effect on large-scale CMB anisotropies comes from late times, and we can set $\eta_{\mathcal{E}} \simeq \eta_m$.

Moreover, since the level of second order tensor modes produced during inflation is tiny, of the order of the slow-roll parameters, we will consider only the post-inflationary evolution.

The evolution equation is obtained by taking the traceless $(i-j)$ component of the Einstein equations for the metric (3.3.25) in the Poisson gauge, and by applying the projector on tensor modes $\left(\mathcal{P}^i{}_k\mathcal{P}^l{}_j - \frac{1}{2}\mathcal{P}^i{}_j\mathcal{P}^l{}_k\right)$ where $\mathcal{P}^i{}_j = \left(\delta^i{}_j - \nabla^{-2}\partial^i\partial_j\right)$. We find

$$\begin{aligned}\chi^{i\prime\prime}_{(2)\,j} + 2\mathcal{H}\chi^{i\prime}_{(2)\,j} - \nabla^2\chi^i_{(2)\,j} =& 4\nabla^{-2}\partial^k\partial_l R^l{}_k\delta^i{}_j + 4\nabla^{-4}\partial^i\partial_j\partial^k\partial_l R^l{}_k + \\ &+ 8\nabla^{-2}\left(\nabla^2 R^i{}_j - \partial^i\partial_k R^k_j - \partial^k\partial_j R^i{}_k\right) ,\end{aligned} \tag{5.4.82}$$

where we have defined

$$\begin{aligned}R^l{}_k \equiv& \partial^l\varphi\partial_k\varphi - \frac{1}{3}(\nabla\varphi)^2\delta^l{}_k + 4\pi G a^2\rho^{(0)}{}_m\left(v^l_{(1)}v^{(1)}{}_k - \frac{1}{3}v^2_{(1)}\delta^l{}_k\right) = \\ =& g^2\left[1 + \frac{2}{3}\frac{E^2(z)f^2(\Omega_m)}{\Omega_{0m}(1+z)^3}\right]\left(\partial^l\varphi_0\partial_k\varphi_0 - \frac{1}{3}(\nabla\varphi_0)^2\delta^l{}_k\right) .\end{aligned} \tag{5.4.83}$$

It is convenient to rewrite the source term on the RHS of eq. (5.4.82) as $s(\eta)\mathcal{T}^i_j(\mathbf{k})$, where

$$s(\eta) = -8g^2\left[1 + \frac{2}{3}\frac{E^2(z)f^2(\Omega_m)}{\Omega_{0m}(1+z)^3}\right] \tag{5.4.84}$$

and

$$\nabla^2\mathcal{T}_{ij}(\mathbf{k}) = \nabla^2\Theta_0\delta_{ij} + \partial_i\partial_j\Theta_0 + 2\left(\partial_i\partial_j\varphi_0\nabla^2\varphi_0 - \partial_i\partial_k\varphi_0\partial^k\partial_j\varphi_0\right) , \tag{5.4.85}$$

with

$$\nabla^2\Theta_0 = -\frac{1}{2}\left[(\nabla^2\varphi_0)^2 - \partial_i\partial_k\varphi_0\partial^i\partial^k\varphi_0\right] . \tag{5.4.86}$$

Now, we find the solution of eq. (5.4.82) for the Fourier transform of $\chi^i_{(2)j}$:

$$\begin{aligned}\chi^i_{(2)j}(\mathbf{k},\eta) =& \mathcal{T}^i_j(\mathbf{k})h(k,\eta) = \\ =& \mathcal{T}^i_j(\mathbf{k})\left[\chi_1(k,\eta)\int_{\eta_m}^{\eta} d\tilde{\eta}\frac{\chi_2(k,\tilde{\eta})}{W}s(\tilde{\eta}) - \chi_2(k,\eta)\int_{\eta_m}^{\eta} d\tilde{\eta}\frac{\chi_1(k,\tilde{\eta})}{W}s(\tilde{\eta})\right] ,\end{aligned} \tag{5.4.87}$$

where χ_1 and χ_2 are the two independent solutions for the gravity-wave equation

$$\chi_1'' + 2\mathcal{H}\chi_i' + k^2\chi_i = 0 \tag{5.4.88}$$

and $W = \chi_1'\chi_2 - \chi_1\chi_2'$ is the corresponding Wronskian. The solutions are exactly known in the limiting cases $\Omega_m \to 1$ and $\Omega_\Lambda \to 1$, but the full solution in a ΛCDM case requires a numerical evaluation.

To compute the multipoles, it is useful to decompose the tensor modes into the $\sigma = +\,,\times$ polarization modes as

$$\chi^i_{(2)j}(\mathbf{k},\eta) = \chi_+(\mathbf{k},\eta)\varepsilon^i_{+j}(\hat{\mathbf{k}}) + \chi_\times(\mathbf{k},\eta)\varepsilon^i_{\times j}(\hat{\mathbf{k}}) \tag{5.4.89}$$

where $\varepsilon^i_{\sigma j}(\hat{\mathbf{k}})$ are the polarization tensors.

Inserting now the expression of the temperature anisotropies (5.4.81) into the definition of the multipoles, eq. (5.4.36), we obtain for the $+$ polarization mode

$$\begin{aligned} a^+_{lm}(\mathbf{k}) =&(-i)^l \int d^2\hat{\mathbf{n}} Y^*_{lm}(\hat{\mathbf{n}}) \left[\frac{1}{4}\int_{\eta_O}^{\eta_m} d\eta \chi'_+(\mathbf{k},\eta)\right] \varepsilon^+_{ij}(\hat{\mathbf{k}}) n^i n^j \times \\ &\times \sum_{l'} i^{l'}(2l'+1) j_{l'}(k(\eta_0-\eta)) P_{l'}(\hat{\mathbf{n}})\,, \end{aligned} \tag{5.4.90}$$

where we have used the Legendre expansion $e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_l i^l(2l+1)j_l(kx)P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{x}})$, and a similar expression holds for the $\times$ polarization mode.

Now we can perform the angular integral for a specific $\hat{\mathbf{k}}$ mode, by choosing the coordinate system such that $\hat{\mathbf{z}} = \hat{\mathbf{k}}$. In fact, statistical isotrpy of observable quantities, like the angular power spectrum and the angle averaged bispectrum, ensures that we can take the sum over the different $\hat{\mathbf{k}}$ modes at the end. With such a coordinate system choice the integral in eq. (5.4.90) gives

$$\begin{aligned} a^+_{lm}(\mathbf{k}) =&(-i)^l \frac{\sqrt{\pi}}{4}\sqrt{2l+1}\sqrt{\frac{(l+2)!}{(l-2)!}}(\delta^2_m + \delta^{-2}_m)\int_{\eta_m}^{\eta_O} d\eta\chi'_+(\mathbf{k},\eta)\times \\ &\times \left[\frac{j_{l+2}(k(\eta_0-\eta))}{(2l+3)(2l+1)} + 2\frac{j_l(k(\eta_0-\eta))}{(2l+3)(2l-1)} + \frac{j_{l-2}(k(\eta_0-\eta))}{(2l+1)(2l-1)}\right] = \\ =&(-i)^l \frac{\sqrt{\pi}}{4}\sqrt{2l+1}\sqrt{\frac{(l+2)!}{(l-2)!}}(\delta^2_m + \delta^{-2}_m)\times \\ &\times \mathcal{T}_+(\mathbf{k})\int_{\eta_m}^{\eta_O} d\eta h'(k,\eta)\frac{j_l(k(\eta_0-\eta))}{[k(\eta_0-\eta)]^2}\,, \end{aligned} \tag{5.4.91}$$

where we have applied the recursion relation $\frac{j_l(x)}{x} = \frac{j_{l-1}(x)+j_{l+1}(x)}{2l+1}$, and $\mathcal{T}+(\mathbf{k})$ is the $+$ component of the amplitude $\mathcal{T}^i_j(\mathbf{k})$. For the $\times$ polarization mode one has to replace $(\delta^2_m + \delta^{-2}_m)$ with $i(\delta^{-2}_m - \delta^2_m)$.

5.4.4 Sachs-Wolfe effect

Starting from the expression of the Sachs-Wolfe effect

$$\frac{\delta^{(2)}T}{T} = \frac{1}{18}\varphi_{\mathcal{E}}^2 - \frac{\mathcal{K}}{10} - \frac{5}{9}(a_{NL} - 1)\varphi_{\mathcal{E}}^2 \tag{5.4.92}$$

where $\mathcal{K} \equiv 10\nabla^{-4}\partial_i\partial^j(\partial^i\varphi_{\mathcal{E}}\partial_j\varphi_{\mathcal{E}}) - \nabla^{-2}\left(\frac{10}{3}\partial^i\varphi_{\mathcal{E}}\partial_i\varphi_{\mathcal{E}}\right)$, the coefficients of the second order transfer function are given by

$$\Delta_l^{0(2)}(k) = \frac{1}{3}j_l(k(\eta_{\mathcal{O}} - \eta_{\hat{\mathbf{e}}}))\,, \tag{5.4.93}$$

$$\Delta_l^{1(2)}(k) = \frac{7}{18}j_l(k(\eta_{\mathcal{O}} - \eta_{\hat{\mathbf{e}}}))\,, \tag{5.4.94}$$

$$\Delta_l^{2(2)}(k) = -\frac{1}{3}j_l(k(\eta_{\mathcal{O}} - \eta_{\hat{\mathbf{e}}}))\,, \tag{5.4.95}$$

$$\Delta_{L_1L_2}(k_1, k_2) = 0\,. \tag{5.4.96}$$

Chapter 6

Conclusions

Measurements of the Cosmic Microwave Background anisotropy are fundamental to the development of our knowledge about the universe. Thanks to the experiments performed in recent years, cosmology has become a precision science, and there is now a general consensus about what is called the standard cosmological model.

Inflation is one of the main pillars of this model, and its generic predictions have been verified by the recent CMB experiments. However, the mechanism by which inflation is attained and, correspondingly, the scenario of generation of cosmological perturbations are not yet fully established.

An observable that can discriminate among these scenarios is the non-Gaussianity of the cosmological perturbations. In order to compare theory with observations, it is important to give definite theoretical predictions for the observable related to the level of non-Gaussianity we expect to measure in the CMB temperature anisotropies, taking into account not only the primordial contribution but all the additional sources of non-linearity in the post-inflationary evolution of the perturbations.

The observational limits obtained up to now don't reach the sensitivity we need to constrain a mild level of non-Gaussianity: in terms of the non-linearity parameter f_{NL}, the actual limits are $-54 < f_{NL} < 114$, with the main uncertainties coming from the subtraction of the foreground. The expected limits with Planck are better, $|f_{NL}| \lesssim 5$, and we can reach a better result, $|f_{NL}| \lesssim 3$, if we add polarization information. Using also independent statistical tools to explore the non-Gaussianity, one could reach the sensitivity we need to detect at least the non-Gaussianity one expects from the post-inflationary evolution, which is of order 1.

In this thesis we have shown how to compute the different contributions to the bispectrum of the CMB temperature anisotropies on large scales, where we have to consider only gravitational effects.

In particular, we have computed the second-order radiation transfer function for large-scale CMB anisotropies, in the case of a ΛCDM universe, taking into account all the relevant effects: in the final expressions it is apparent that the primordial contribution is disentangled from the gravitational non-linearities.

To this end, we have made use of a non-perturbative formalism to write the evolution equations and a general expression for large-scale temperature anisotropies for scalar perturbations in the Poisson gauge, which represent the original contribution of this work. The formalism can be extended to include tensor modes, whose (integrated) contribution is non-negligible on large scales.

Our approach can help to study other signals of non-Gaussianity, for example the trispectrum, which would require an analysis up to third order in the perturbations, and higher order connected correlation functions, or a non-perturbative quantity such as the probability density function of the perturbations.

Appendix A

Einstein tensor in the non-perturbative approach

Metric

$$ds^2 = a^2(\eta)\left[-e^{2\Phi}d\eta^2 + e^{-2\Psi}\delta_{ij}dx^i dx^j\right] \tag{A.1}$$

Cristoffel symbols

$$\Gamma^0_{00} = \mathcal{H} + \Phi' \tag{A.2}$$
$$\Gamma^0_{0i} = \partial_i\Phi \tag{A.3}$$
$$\Gamma^0_{ij} = e^{-2(\Phi+\Psi)}\left(\mathcal{H} - \Psi'\right)\delta_{ij} \tag{A.4}$$
$$\Gamma^i_{00} = e^{2(\Phi+\Psi)}\partial^i\Phi \tag{A.5}$$
$$\Gamma^i_{0j} = \left(\mathcal{H} - \Psi'\right)\delta^i_j \tag{A.6}$$
$$\Gamma^i_{jk} = -(\partial_j\Psi)\delta^i_k - (\partial_k\Psi)\delta^i_j + (\partial^i\Psi)\delta_{jk} \tag{A.7}$$

Ricci tensor

$$R_{00} = e^{2(\Phi+\Psi)}\left[\partial^l\Phi\partial_l\Phi + \nabla^2\Phi - \partial^l\Phi\partial_l\Psi\right] + \\ + 3\left[-\mathcal{H}' + \mathcal{H}\left(\Phi' + \Psi'\right) - \Phi'\Psi' - \Psi'^2 + \Psi''\right] \tag{A.8}$$

$$R^0_{\ 0} = \frac{e^{-2\Phi}}{a^2}3\left[\mathcal{H}' - \mathcal{H}\left(\Phi' + \Psi'\right) + \Phi'\Psi' + \Psi'^2 - \Psi''\right] + \\ + \frac{e^{2\Psi}}{a^2}\left[\partial^l\Phi\partial_l\Psi - \partial^l\Phi\partial_l\Phi - \nabla^2\Phi\right] \tag{A.9}$$

$$R_{0i} = 2\partial_i\Psi' + 2\left(\mathcal{H} - \Psi'\right)\partial_i\Phi \tag{A.10}$$

$$R^i_{\ 0} = \frac{e^{2\Psi}}{a^2} 2\left[\partial^i\Psi' + \left(\mathcal{H} - \Psi'\right)\partial^i\Phi\right] \tag{A.11}$$

$$R^0_{\ i} = -\frac{e^{-2\Phi}}{a^2} 2\left[\partial_i\Psi' + \left(\mathcal{H} - \Psi'\right)\partial_i\Phi\right] \tag{A.12}$$

$$\begin{aligned} R_{ij} =& e^{-2(\Phi+\Psi)}\left[\left(\mathcal{H} - \Psi'\right)\left(2\mathcal{H} - 3\Psi' - \Phi'\right) + \mathcal{H}' - \Psi''\right]\delta_{ij} + \\ &+ \partial_i\partial_j\Psi - \partial_i\partial_j\Phi + \nabla^2\Psi\delta_{ij} + \partial_i\Psi\partial_j\Psi - \partial^l\Psi\partial_l\Psi\delta_{ij} - \partial_i\Phi\partial_j\Phi + \\ &- \partial_i\Phi\partial_j\Psi - \partial_i\Psi\partial_j\Phi + \partial^l\Phi\partial_l\Psi\delta_{ij} \end{aligned} \tag{A.13}$$

$$\begin{aligned} R^i_{\ j} =& \frac{e^{-2\Phi}}{a^2}\left[\left(\mathcal{H} - \Psi'\right)\left(2\mathcal{H} - 3\Psi' - \Phi'\right) + \mathcal{H}' - \Psi''\right]\delta^i_j + \\ &+ \frac{e^{2\Psi}}{a^2}\Big[\partial^i\partial_j\Psi - \partial^i\partial_j\Phi + \nabla^2\Psi\delta^i_j + \partial^i\Psi\partial_j\Psi - \partial^l\Psi\partial_l\Psi\delta^i_j + \\ &-\partial^i\Phi\partial_j\Phi - \partial^i\Phi\partial_j\Psi - \partial^i\Psi\partial_j\Phi + \partial^l\Phi\partial_l\Psi\delta^i_j\Big] \end{aligned} \tag{A.14}$$

$$\begin{aligned} R^l_{\ l} =& \frac{e^{-2\Phi}}{a^2} 3\left[\left(\mathcal{H} - \Psi'\right)\left(2\mathcal{H} - 3\Psi' - \Phi'\right) + \mathcal{H}' - \Psi''\right] + \\ &+ \frac{e^{2\Psi}}{a^2}\left[4\nabla^2\Psi - \nabla^2\Phi - 2\partial^l\Psi\partial_l\Psi - \partial^l\Phi\partial_l\Phi + \partial^l\Phi\partial_l\Psi\right] \end{aligned} \tag{A.15}$$

Ricci scalar

$$\begin{aligned} R =& \frac{e^{-2\Phi}}{a^2} 6\left[\left(\mathcal{H} - \Psi'\right)\left(\mathcal{H} - \Phi' - 2\Psi'\right) + \mathcal{H}' - \Psi''\right] + \\ &+ \frac{e^{2\Psi}}{a^2}\left[2\partial^l\Phi\partial_l\Psi - 2\partial^l\Phi\partial_l\Phi - 2\nabla^2\Phi + 4\nabla^2\Psi - 2\partial^l\Psi\partial_l\Psi\right] \end{aligned} \tag{A.16}$$

Einstein Tensor

$$G^0_{\ 0} = -\frac{e^{-2\Phi}}{a^2} 3\left(\mathcal{H} - \Psi'\right)^2 + \frac{e^{2\Psi}}{a^2}\left[\partial^l\Psi\partial_l\Psi - 2\nabla^2\Psi\right] \tag{A.17}$$

$$G^i_{\ 0} = R^i_{\ 0}\,, \quad G^0_{\ i} = R^0_{\ i} \tag{A.18}$$

$$\begin{aligned} G^i_{\ j} - \frac{1}{3}G^l_{\ l}\delta^i_j =& \frac{e^{2\Psi}}{a^2}\Big[\partial^i\partial_j\Psi - \frac{1}{3}\nabla^2\Psi\delta^i_j - \partial^i\partial_j\Phi + \frac{1}{3}\nabla^2\Phi + \partial^i\Psi\partial_j\Psi - \frac{1}{3}\partial^l\Psi\partial_l\Psi\delta^i_j + \\ &-\partial^i\Phi\partial_j\Phi + \frac{1}{3}\partial^l\Phi\partial_l\Phi\delta^i_j - \partial^i\Phi\partial_j\Psi - \partial^i\Psi\partial_j\Phi + \frac{2}{3}\partial^l\Phi\partial_l\Psi\delta^i_j\Big] \end{aligned} \tag{A.19}$$

$$
\begin{aligned}
{G^l}_l =& \frac{e^{-2\Phi}}{a^2} 3 \left[\left(\mathcal{H} - \Psi' \right) \left(-\mathcal{H}' + 3\Psi' + 2\Phi' \right) - 2\mathcal{H}' + 2\Psi'' \right] + \\
& + \frac{e^{2\Psi}}{a^2} \left[2\nabla^2 \left(\Phi - \Psi \right) + \partial^l \Psi \partial_l \Psi + 2\partial^l \Phi \partial_l \Phi - \partial^l \Phi \partial_l \Psi \right]
\end{aligned}
\tag{A.20}
$$

Appendix B

Einstein equations up to second order

We now expand up to second order the fully non linear evolution equations we have derived in Sec. (5.3), and we specialize them to the case of matter plus radiation and matter plus cosmological constant

Background equations

(0-0)

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \left(\rho_m^{(0)} + \rho_\gamma^{(0)}\right) + a^2\Lambda \tag{B.1}$$

(i-i)

$$\mathcal{H}^2 + 2\mathcal{H}' = a^2\Lambda - 8\pi G a^2 \frac{1}{3}\rho_\gamma^{(0)} \tag{B.2}$$

First-order equations

(0-0)

$$\mathcal{H}^2\phi^{(1)} + \mathcal{H}'\psi^{(1)} = -\frac{4\pi G}{3} a^2 \left(\delta^{(1)}\rho_m + \delta^{(1)}\rho_\gamma\right) \tag{B.3}$$

(0-i)

$$\partial_i\psi^{(1)\prime} + \mathcal{H}\partial_i\phi^{(1)} = 4\pi G a v_i^{(1)} \left(\delta^{(1)}\rho_m + \frac{4}{3}\delta^{(1)}\rho_\gamma\right) \tag{B.4}$$

(i-i)

$$2\psi^{(1)\prime\prime} + 4\mathcal{H}'\phi^{(1)} + 2\mathcal{H}\phi^{(1)\prime} + 4\mathcal{H}\psi^{(1)\prime} + 2\mathcal{H}^2\phi^{(1)} = 8\pi G a^2 \delta^{(1)}\rho_\gamma \tag{B.5}$$

traceless (i-j)

$$\nabla^4 \left(\psi^{(1)} - \phi^{(1)} \right) = 0 \tag{B.6}$$

Second-order equations

(0-0)

$$\mathcal{H} \left(\psi^{(2)\prime} + \mathcal{H}\phi^{(2)} \right) + 4\mathcal{H}\psi^{(1)}\psi^{(1)\prime} - (\psi^{(1)\prime})^2 - 4\mathcal{H}\phi^{(1)}\psi^{(1)\prime} - 4\mathcal{H}^2(\phi^{(1)})^2 =$$
$$= -\frac{8\pi G}{3} a^2 \left(\delta^{(2)}\rho_m + \delta^{(2)}\rho_\gamma \right) \tag{B.7}$$

(i-i)

$$\mathcal{H}^2 \left(\phi^{(2)} - 4\phi^{(1)2} \right) - 8\mathcal{H}\phi^{(1)}\psi^{(1)\prime} + 2\mathcal{H}\psi^{(2)\prime} + 8\mathcal{H}\psi^{(1)}\psi^{(1)\prime} - 8\mathcal{H}\phi^{(1)}\phi^{(1)\prime} +$$
$$+ \mathcal{H}\phi^{(2)\prime} - 3(\psi^{(1)\prime})^2 - 2\phi^{(1)\prime}\psi^{(1)\prime} + 2\mathcal{H}'\phi^{(2)} - 8\mathcal{H}'\phi^{(1)2} - 4\phi^{(1)}\psi^{(1)\prime\prime} +$$
$$+ \psi^{(2)\prime\prime} + 4(\psi^{(1)\prime})^2 + 4\psi^{(1)}\psi^{(1)\prime\prime} - \frac{4}{3}\psi^{(1)}\nabla^2 \left(\psi^{(1)} - \phi^{(1)} \right) - \frac{1}{3}\nabla^2 \left(\psi^{(2)} - \phi^{(2)} \right) +$$
$$- \frac{2}{3}\nabla^2 \left(\psi^{(1)2} + \phi^{(1)2} \right) + \frac{1}{3}(\partial_i\psi^{(1)})^2 + \frac{2}{3}(\partial_i\phi^{(1)})^2 - \frac{2}{3}\partial_i\psi^{(1)}\partial^i\psi^{(1)} =$$
$$= \frac{1}{6\pi G a^2 \rho_m^{(0)}} \left[\partial_i \left(\psi^{(1)\prime} + \mathcal{H}\phi^{(1)} \right) \right]^2 \tag{B.8}$$

traceless (i-j)

$$\frac{1}{2} \left(\partial^i\partial_j - \frac{1}{3}\delta^i{}_j\nabla^2 \right) \left(\psi^{(2)} - \phi^{(2)} \right) + \left(\partial^i\partial_j - \frac{1}{3}\delta^i{}_j\nabla^2 \right) \left[(\psi^{(1)})^2 + (\phi^{(1)})^2 \right] +$$
$$+ \partial^i\psi^{(1)}\partial_j\psi^{(1)} - \frac{1}{3}(\partial_l\psi^{(1)})^2\delta^i{}_j - \partial^i\phi^{(1)}\partial_j\phi^{(1)} + \frac{1}{3}(\partial_l\phi^{(1)})^2\delta^i{}_j +$$
$$- \partial^i\phi^{(1)}\partial_j\psi^{(1)} - \partial^i\psi^{(1)}\partial_j\phi^{(1)} + \frac{2}{3}\delta^i{}_j(\partial^l\psi^{(1)})(\partial_l\phi^{(1)}) =$$
$$= \frac{1}{2\pi G a^2} \frac{1}{\rho_m^{(0)} + \frac{4}{3}\rho_\gamma^{(0)}} \left\{ \partial^i\psi^{(1)\prime}\partial_j\psi^{(1)\prime} - \frac{1}{3}(\partial_l\psi^{(1)\prime})\delta^i{}_j + \mathcal{H}^2 \left[\partial^i\phi^{(1)}\partial_j\phi^{(1)} - \frac{1}{3}(\partial_l\phi^{(1)})\delta^i{}_j \right] + \right.$$
$$\left. + \mathcal{H} \left[\partial^i\phi^{(1)}\partial_j\psi^{(1)\prime} + \partial^i\psi^{(1)\prime}\partial_j\phi^{(1)} - \frac{2}{3}\delta^i{}_j(\partial^l\psi^{(1)\prime})(\partial_l\phi^{(1)}) \right] \right\} \tag{B.9}$$

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